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# ON THE WAVE EQUATION WITH HYPERBOLIC DYNAMICAL BOUNDARY CONDITIONS, INTERIOR AND BOUNDARY DAMPING AND SOURCE

ENZO VITILLARO

Abstract. The aim of this paper is to study the problem

$$\begin{cases} u_{tt} - \Delta u + P(x, u_t) = f(x, u) & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ u_{tt} + \partial_\nu u - \Delta_\Gamma u + Q(x, u_t) = g(x, u) & \text{on } (0, \infty) \times \Gamma_1, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{in } \overline{\Omega}, \end{cases}$$

where $\Omega$ is a open bounded subset of $\mathbb{R}^N$ with $C^1$ boundary ($N \geq 2$), $\Gamma = \partial\Omega$, $(\Gamma_0, \Gamma_1)$ is a measurable partition of $\Gamma$, $\Delta_\Gamma$ denotes the Laplace–Beltrami operator on $\Gamma$, $\nu$ is the outward normal to $\Omega$, and the terms $P$ and $Q$ represent nonlinear damping terms, while $f$ and $g$ are nonlinear subcritical perturbations.

In the paper a local Hadamard well–posedness result for initial data in the natural energy space associated to the problem is given. Moreover, when $\Omega$ is $C^2$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, the regularity of solutions is studied. Next a blow–up theorem is given when $P$ and $Q$ are linear and $f$, $g$ are superlinear sources. Finally a dynamical system is generated when the source parts of $f$ and $g$ are at most linear at infinity, or they are dominated by the damping terms.

## 1. Introduction and main result

We deal with the evolution problem consisting of the wave equation posed in a bounded regular open subset of $\mathbb{R}^N$, supplied with a second order dynamical boundary condition of hyperbolic type, in presence of interior and/or boundary damping terms and sources. More precisely we consider the initial–and–boundary value problem

$$\begin{cases} u_{tt} - \Delta u + P(x, u_t) = f(x, u) & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ u_{tt} + \partial_\nu u - \Delta_\Gamma u + Q(x, u_t) = g(x, u) & \text{on } (0, \infty) \times \Gamma_1, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{in } \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega$ is a open bounded subset of $\mathbb{R}^N$ ($N \geq 2$) with $C^1$ boundary (see [33]). We denote $\Gamma = \partial\Omega$ and we assume $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_1$ being relatively open

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on $\Gamma$ (or equivalently $\overline{\Gamma_0} = \Gamma_0$). Moreover, denoting by $\sigma$ the standard Lebesgue hypersurface measure on $\Gamma$, we assume that $\sigma(\overline{\Gamma}_0 \cap \overline{\Gamma}_1) = 0$. These properties of $\Omega$, $\Gamma_0$ and $\Gamma_1$ will be assumed, without further comments, throughout the paper. In Section 5 we shall restrict to open bounded subsets with $C^2$ boundary and to partitions such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Moreover $u = u(t,x)$, $t \geq 0$, $x \in \Omega$, $\Delta = \Delta_x$ denotes the Laplace operator with respect to the space variable, while $\Delta_\Gamma$ denotes the Laplace–Beltrami operator on $\Gamma$ and $\nu$ is the outward normal to $\Omega$. The terms $P$ and $Q$ represent nonlinear damping terms, i.e. $P(x,v)v \geq 0$, $Q(x,v)v \geq 0$, the cases $P \equiv 0$ and $Q \equiv 0$ being specifically allowed, while $f$ and $g$ represent nonlinear source, or sink, terms. The specific assumptions on them will be introduce d later on.

Problems with kinetic boundary conditions, that is boundary conditions involving $u_{tt}$, on $\Gamma$ or on a part of it, naturally arise in several physical applications. A one dimensional model was studied by several authors to describe transversal small oscillations of an elastic rod with a tip mass on one endpoint, while the other one is pinched. See [4, 20, 21, 34, 44].

A two dimensional model introduced in [31] deals with a vibrating membrane of surface density $\mu$, subject to a tension $T$, both taken constant and normalized here for simplicity. If $u(t,x)$, $x \in \Omega \subset \mathbb{R}^2$ denotes the vertical displacement from the rest state, then (after a standard linear approximation) $u$ satisfies the wave equation $u_{tt} - \Delta u = 0$, $(t,x) \in \mathbb{R} \times \Omega$. Now suppose that a part $\Gamma_0$ of the boundary is pinched, while the other part $\Gamma_1$ carries a constant linear mass density $m > 0$ and it is subject to a linear tension $\tau$. A practical example of this situation is given by a drumhead with a hole in the interior having a thick border, as common in bass drums. One linearly approximates the force exerted by the membrane on the boundary with $-\partial_\nu u$. The boundary condition thus reads as $mu_{tt} + \partial_\nu u - \tau \Delta_{\Gamma_1} u = 0$. In the quoted paper the case $\Gamma_0 = \emptyset$ and $\tau = 0$ was studied, while here we consider the more realistic case $\Gamma_0 \neq \emptyset$ and $\tau > 0$, with $\tau$ and $m$ normalized for simplicity. We would like to mention that this model belongs to a more general class of models of Lagrangian type involving boundary energies, as introduced for example in [25].

A three dimensional model involving kinetic dynamical boundary conditions comes out from [27], where a gas undergoing small irrotational perturbations from rest in a domain $\Omega \subset \mathbb{R}^3$ is considered. Normalizing the constant speed of propagation, the velocity potential $\phi$ of the gas (i.e. $-\nabla\phi$ is the particle velocity) satisfies the wave equation $\phi_{tt} - \Delta\phi = 0$ in $\mathbb{R} \times \Omega$. Each point $x \in \partial\Omega$ is assumed to react to the excess pressure of the acoustic wave like a resistive harmonic oscillator or spring, that is the boundary is assumed to be locally reacting (see [45, pp. 259–264]). The normal displacement $\delta$ of the boundary into the domain then satisfies $m\delta_{tt} + d\delta_t + k\delta + \rho\phi_t = 0$, where $\rho > 0$ is the fluid density and $m, d, k \in C(\partial\Omega)$, $m, k > 0$, $d \geq 0$. When the boundary is nonporous one has $\delta_t = \partial_\nu \phi$ on $\mathbb{R} \times \partial\Omega$, so the boundary condition reads as $m\delta_{tt} + d\partial_\nu\phi + k\delta + \rho\phi_t = 0$. In the particular case $m = k$ and $d = \rho$ (see [27, Theorem 2]) one proves that $\phi_{|\Gamma} = \delta$, so the boundary condition reads as $m\phi_{tt} + d\partial_\nu\phi + k\phi + \rho\phi_t = 0$, on $\mathbb{R} \times \partial\Omega$. Now, if one consider s the case in which the boundary is not locally reacting, as in [11], one has to had a Laplace–Beltrami term so getting an hyperbolic dynamical boundary condition like the one in (1.1).

Several papers in the literature deal with the wave equation with kinetic boundary conditions. This fact is even more evident if one takes into account that, plugging the equation in (1.1) into the boundary condition, we can rewrite it as $\Delta u + \partial_\nu u - \Delta_\Gamma u + Q(x, u_t) + P(x, u_t) = f(x, u) + g(x, u)$. Such a condition is usually called a *generalized Wentzell boundary condition*, at least when nonlinear perturbations are not present. We refer to [46], where abstract semigroup techniques are applied to dissipative wave equations, and to [23, 24, 61, 67, 68]. All of them deal either with the case $\tau = 0$ or with linear problems.

Here we shall consider this type of kinetic boundary condition in connection with nonlinear boundary damping and source terms. These terms have been considered by several authors, but mainly in connection with first order dynamical boundary conditions. See [5, 6, 13, 14, 15, 17, 18, 19, 37, 64, 65]. The competition between interior damping and source terms is methodologically related to the competition between boundary damping and source and it possesses a large literature as well. See [7, 28, 39, 48, 49, 52, 63].

Problem (1.1) has been recently introduced by the author in [66], dealing with a preliminary analysis of (1.1) in the particular the case $P = 0$, $f = 0$, $Q = |u_t|^{\mu-2}u_t$, $g = |u|^{q-2}u$, $\mu > 1$, $q \geq 2$. When $\Omega$ is $C^2$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, so $\Gamma$ is disconnected, both $Q$ and $g$ are subcritical with respect to the Sobolev embedding on $\Gamma$, and $u_0 \in H^2(\Omega)$, $u_{0|\Gamma_1} \in H^2(\Gamma_1)$, $u_{1,\Omega} \in H^1(\Omega)$, $u_{1,\Gamma_1} = u_{1,\Omega_{\Gamma_1}} \in H^1(\Gamma_1)$, an existence and uniqueness result is proved. Moreover a linear problem strongly related to (1.1) has also been recently studied in [32], dealing with analiticity or Gevrey classification for the generated linear semigroup, and in [26], dealing with regularity and stability.

The aim of the present paper is to substantially generalize the analysis made in [66] in several directions. At first we want to treat in an unified framework interior and/or internal source and damping terms, each of which can vanish identically (the alternative being the study of several different problems). At second we want to include supercritical boundary (as well as internal) damping terms. Next we want to allow $\Gamma$ to be connected and just $C^1$ . Moreover we want to consider initial data in the natural energy space related to (1.1) and thus weak solutions of it. Finally we plan to study local Hadamard well-posedness. Several technical problems, which were not present in [66], makes the analysis more involved. To best illustrate our results we consider, in this section, the simplified version of (1.1)

$$
\begin{cases}
u_{tt} - \Delta u + \alpha(x) P_0(u_t) = f_0(u) & \text{in } (0,\infty) \times \Omega, \\
u = 0 & \text{on } (0,\infty) \times \Gamma_0, \\
u_{tt} + \partial_\nu u - \Delta_\Gamma u + \beta(x) Q_0(u_t) = g_0(u) & \text{on } (0,\infty) \times \Gamma_1, \\
u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & \text{in } \overline{\Omega},
\end{cases}
\tag{1.2}
$$

where $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma_1)$, $\alpha, \beta \geq 0$, conventionally taking $g_0 \equiv 0$ when $\Gamma_1 = \emptyset$, and the following properties are assumed:

(I) $P_0$ and $Q_0$ are continuous and monotone increasing in $\mathbb{R}$, $P_0(0) = Q_0(0) = 0$, and there are $m, \mu > 1$ such that

$$0 < \liminf_{|v|\to\infty} \frac{|P_0(v)|}{|v|^{m-1}} \le \limsup_{|v|\to\infty} \frac{|P_0(v)|}{|v|^{m-1}} < \infty, \quad \liminf_{|v|\to 0} \frac{|P_0(v)|}{|v|^{m-1}} > 0,$$
$$0 < \liminf_{|v|\to\infty} \frac{|Q_0(v)|}{|v|^{\mu-1}} \le \limsup_{|v|\to\infty} \frac{|Q_0(v)|}{|v|^{\mu-1}} < \infty, \quad \liminf_{|v|\to 0} \frac{|Q_0(v)|}{|v|^{\mu-1}} > 0;$$

(II) $f_0, g_0 \in C^{0,1}_{\text{loc}}(\mathbb{R})$ and there are $p, q \ge 2$ such that $|f_0'(u)| = O(|u|^{p-2})$ and $|g_0'(u)| = O(|u|^{q-2})$ as $|u| \to \infty$.

Our model nonlinearities satisfying (I–II) are given by

$$\begin{cases} P_0(v) = P_1(v) := a|v|^{\widetilde{m}-2}v + |v|^{m-2}v, & 1 < \widetilde{m} \le m, \quad a \ge 0, \\ Q_0(v) = Q_1(v) := b|v|^{\widetilde{\mu}-2}v + |v|^{\mu-2}v, & 1 < \widetilde{\mu} \le \mu, \quad b \ge 0, \\ f_0(u) = f_1(u) := \widetilde{\gamma}|u|^{\widetilde{p}-2}u + \gamma|u|^{p-2}u + c_1, & 2 \le \widetilde{p} \le p, \quad \widetilde{\gamma}, \gamma, c_1 \in \mathbb{R}, \\ g_0(u) = g_1(u) := \widetilde{\delta}|u|^{\widetilde{q}-2}u + \delta|u|^{q-2}u + c_2, & 2 \le \widetilde{q} \le q, \quad \widetilde{\delta}, \delta, c_2 \in \mathbb{R}. \end{cases} \tag{1.3}$$

We introduce some basic notation. In the sequel we shall identify $L^2(\Gamma_1)$ with its isometric image in $L^2(\Gamma)$, that is

$$L^2(\Gamma_1) = \{u \in L^2(\Gamma) : u = 0 \text{ a.e. on } \Gamma_0\}. \tag{1.4}$$

We set, for $\rho \in [1, \infty)$ and $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma_1)$, $\alpha, \beta \ge 0$, the Banach spaces

$$L^{2,\rho}_\alpha(\Omega) = \{u \in L^2(\Omega) : \alpha^{1/\rho}u \in L^\rho(\Omega)\}, \qquad \|\cdot\|_{2,\rho,\alpha} = \|\cdot\|_2 + \|\alpha^{1/\rho}\cdot\|_\rho,$$
$$L^{2,\rho}_\beta(\Gamma_1) = \{u \in L^2(\Gamma_1) : \beta^{1/\rho}u \in L^\rho(\Gamma_1)\}, \quad \|\cdot\|_{2,\rho,\beta} = \|\cdot\|_{2,\Gamma_1} + \|\beta^{1/\rho}\cdot\|_{\rho,\Gamma_1},$$

where $\|\cdot\|_\rho := \|\cdot\|_{L^\rho(\Omega)}$ and $\|\cdot\|_{\rho,\Gamma_1} := \|\cdot\|_{L^\rho(\Gamma_1)}$ [1].

We denote by $u_{|\Gamma}$ the trace on $\Gamma$ of any $u \in H^1(\Omega)$, and by $u_{|\Gamma_i}$ its restriction to $\Gamma_i$, $i = 0, 1$. Moreover we introduce the Hilbert spaces $H^0 = L^2(\Omega) \times L^2(\Gamma_1)$,

$$H^1 = \{(u, v) \in H^1(\Omega) \times H^1(\Gamma) : v = u_{|\Gamma}, v = 0 \text{ on } \Gamma_0\}, \tag{1.5}$$

with the topology inherited from the products. For the sake of simplicity we shall identify, when useful, $H^1$ with its isomorphic counterpart $\{u \in H^1(\Omega) : u_{|\Gamma} \in H^1(\Gamma) \cap L^2(\Gamma_1)\}$, through the identification $(u, u_{|\Gamma}) \mapsto u$, so we shall write, without further mention, $u \in H^1$ for functions defined on $\Omega$. Moreover we shall drop the notation $u_{|\Gamma}$, when useful, so we shall write $\|u\|_{2,\Gamma}$, $\int_\Gamma u$, and so on, for elements of $H^1$. We also introduce, for $\alpha$ and $\beta$ as before and $\rho, \theta \in [1, \infty]$, the Banach space

$$H^{1,\rho,\theta}_{\alpha,\beta} = H^1 \cap [L^{2,\rho}_\alpha(\Omega) \times L^{2,\theta}_\beta(\Gamma_1)], \quad \|\cdot\|_{H^{1,\rho,\theta}_{\alpha,\beta}} = \|\cdot\|_{H^1} + \|\cdot\|_{L^{2,\rho}_\alpha(\Omega)\times L^{2,\theta}_\beta(\Gamma_1)}. \tag{1.6}$$

Next, when $\Omega$ is $C^2$ and $\rho \in [1, \infty]$, we denote

$$W^{2,\rho} = [W^{2,\rho}(\Omega) \times W^{2,\rho}(\Gamma)] \cap H^1, \qquad \text{and} \qquad H^2 = W^{2,2}, \tag{1.7}$$

[1] it would appear simpler to set $L^{2,\rho}_\alpha(\Omega) = L^2(\Omega) \cap L^\rho(\Omega, \lambda_\alpha)$, but unfortunately when $\alpha$ vanishes in a set of positive measure that is wrong, since the equivalence classes in the two intersecting spaces are different, as it is clear in the extreme case $\alpha \equiv 0$.

endoweed with the norm inherited from the product. Finally we set $r_\Omega$ and $r_\Gamma$ to respectively be the critical exponents of the Sobolev embeddings [2] $H^1(\Omega) \hookrightarrow L^s(\Omega)$ and $H^1(\Gamma) \hookrightarrow L^s(\Gamma)$, that is

$$r_\Omega = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 2, \end{cases} \qquad r_\Gamma = \begin{cases} \dfrac{2(N-1)}{N-3} & \text{if } N \geq 4, \\ \infty & \text{if } N = 2, 3. \end{cases}$$

The first aim of the paper is to show that the problem (1.2) is locally well-posed in the Hadamard sense in the phase space $H^1 \times H^0$ when $f_0$ and $g_0$ are subcritical in the sense of semigroups.

**Theorem 1.1** (**Local well–posedness in $\boldsymbol{H^1 \times H^0}$**). *If (I–II) hold and*

$$2 \leq p \leq 1 + r_\Omega/2, \qquad 2 \leq q \leq 1 + r_\Gamma/2, \tag{1.8}$$

*then the following conclusions hold.*

(i) *For any $(u_0, u_1) \in H^1 \times H^0$ problem (1.2) has a unique maximal weak solution $u$ in $[0, T_{max})$, that is*

$$u = (u, u_{|\Gamma}) \in L^\infty_{loc}([0, T_{max}); H^1) \cap W^{1,\infty}_{loc}([0, T_{max}); H^0), \tag{1.9}$$

$$u' = (u_t, (u_{|\Gamma})_t) \in L^m_{loc}([0, T_{max}); L^{2,m}_\alpha(\Omega)) \times L^\mu_{loc}([0, T_{max}); L^{2,\mu}_\beta(\Gamma_1)), \tag{1.10}$$

*which satisfies (1.2) in a distribution sense to be specified later on;*

(ii) *$u$ enjoys the regularity*

$$u \in C([0, T_{max}); H^1) \cap C^1([0, T_{max}); H^0) \tag{1.11}$$

*and satisfies, for $0 \leq s \leq t < T_{max}$, the energy identity* [3]

$$\frac{1}{2}\left[\int_\Omega u_t^2(\tau) + \int_{\Gamma_1} (u_{|\Gamma})_t^2(\tau) + \int_\Omega |\nabla u(\tau)|^2 + \int_{\Gamma_1} |\nabla_\Gamma u(\tau)|_\Gamma^2\right]_s^t + \int_s^t\int_\Omega \alpha P_0(u_t)u_t$$
$$+ \int_s^t \int_{\Gamma_1} \beta Q_0((u_{|\Gamma})_t))(u_{|\Gamma})_t = \int_s^t \int_\Omega f_0(u)u_t + \int_s^t \int_{\Gamma_1} g_0(u)(u_{|\Gamma})_t;$$

(iii) *if $T_{max} < \infty$ then*

$$\lim_{t \to T_{max}^-} \|u(t)\|_{H^1(\Omega)} + \|u(t)\|_{H^1(\Gamma)} + \|u_t(t)\|_{L^2(\Omega)} + \|(u_{|\Gamma})_t(t)\|_{L^2(\Gamma_1)} = \infty; \tag{1.12}$$

(iv) *if $u_{0n} \to u_0$ in $H^1$, $u_{1n} \to u_1$ in $H^0$ and we respectively denote by $u_n \in C([0, T^n_{max}); H^1)$ and $u \in C([0, T_{max}); H^1)$ the weak maximal solutions of problem (1.2) corresponding to initial data $(u_{0n}, u_{1n})$ and $(u_0, u_1)$, we have $T_{max} \leq \liminf_n T^n_{max}$ and, for any $T \in (0, T_{max})$,*

$$u_n \to u \quad \text{in } C([0,T]; H^1) \cap C^1([0,T]; H^0).$$

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[2] with the well–known exceptions for $r_\Omega$ when $N = 2$ and for $r_\Gamma$ when $N = 3$. The embedding $H^1(\Gamma) \hookrightarrow L^{r_\Gamma}(\Gamma)$ is standard in the $C^\infty$ setting, see for example [35, Theorem 2.6 p.32], and one easily sees that the proof extends to $C^1$ manifolds without changes.

[3] $\nabla_\Gamma$ denotes the Riemannian gradient on $\Gamma$ and $|\cdot|_\Gamma$, the norm associated to the Riemannian scalar product on the tangent bundle of $\Gamma$. See Section 2.

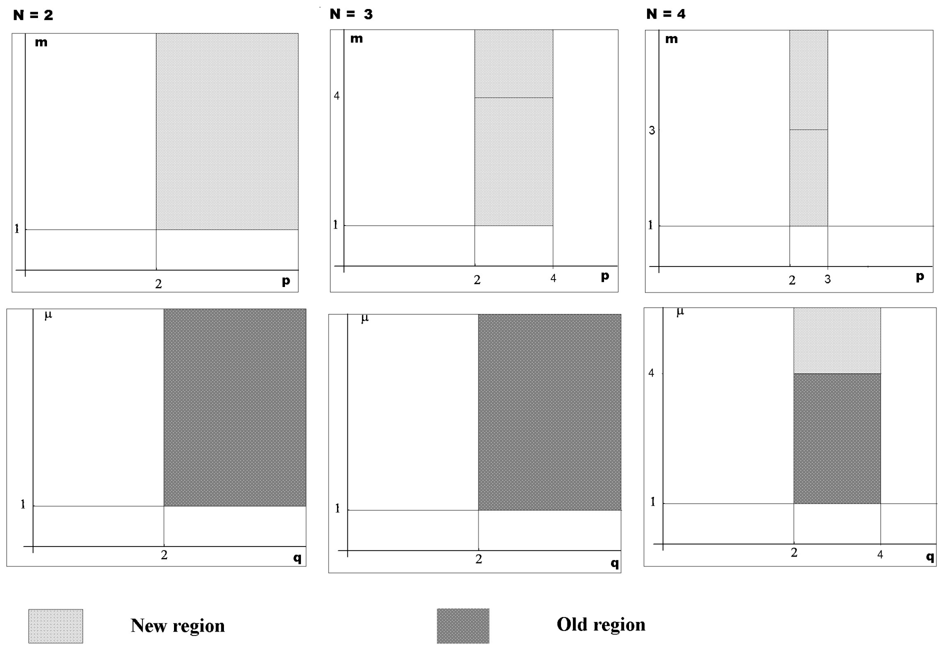


FIGURE 1. The old region is the parameter range treated in [66], while the new region is the range covered only by Theorem 1.1.

*Remark* 1.1. Figure 1 illustrates the parameter ranges covered by Theorem 1.1 and by [66, Theorem 1] in dimensions $N = 2, 3, 4$. Theorem 1.1 is optimal when $N = 2$, while when $N = 3$ assumption (1.8) imposes a severe restriction of the growth of the internal source/sink. When $N = 4$ the restriction concerns both sources/sinks and is even more severe. To relax assumption (1.8) requires a specific analysis which is outside the aim of the present paper. On the other hand there is no restriction on the growth of the damping terms, improving the analysis in [66].

As a simple byproduct of the arguments used to prove Theorem 1.1 we get a well–posedness result in a stronger (when $m > r_\Omega$ or $\mu > r_\Gamma$) topology provided $P_0$ and $Q_0$ satisfy the further assumption, trivially satisfied by $P_1$, $Q_1$ in (1.3),

(III) $\liminf_{|v|\to\infty} \frac{|P_0'(v)|}{|v|^{m-2}} > 0$ if $m > r_\Omega$, $\liminf_{|v|\to\infty} \frac{|Q_0'(v)|}{|v|^{\mu-2}} > 0$ if $\mu > r_\Gamma$.

**Theorem 1.2** (**Local Hadamard well–posedness in $\boldsymbol{H^{1,\rho,\theta}_{\alpha,\beta} \times H^0}$**). *If (I–III) and* (1.8) *hold then, for any couple of exponents*

$$(\rho, \theta) \in [r_\Omega, \max\{r_\Omega, m\}] \times [r_\Gamma, \max\{r_\Gamma, \mu\}] \tag{1.13}$$

*and any* $(u_0, u_1) \in H^{1,\rho,\theta}_{\alpha,\beta} \times H^0$, *the weak solution* $u$ *of problem* (1.2) *enjoys the further regularity*

$$u \in C([0, T_{max}); H^{1,\rho,\theta}_{\alpha,\beta}). \tag{1.14}$$

*Moreover, if* $u_{0n} \to u_0$ *in* $H^{1,\rho,\theta}_{\alpha,\beta}$, $u_{1n} \to u_1$ *in* $H^0$, *and* $u_n$, $u$ *are as in Theorem 1.1, then*

$$u_n \to u \text{ in } C([0, T]; H^{1,\rho,\theta}_{\alpha,\beta}) \text{ for any } T \in (0, T_{max}).$$

Theorems 1.1 and 1.2 can be easily extended to more general second order uniformly elliptic linear operators, both in $\Omega$ and $\Gamma$, under suitable regularity assumptions on the coefficients. Here we prefer to deal with the Laplace and Laplace–Beltrami operators for the sake of clearness. The proof will rely on nonlinear semigroup theory (see [10] and [53]), and in particular on [19, Theorem 7.2, Appendix], and on an easy consequence of the approach used there, which is outlined, for the reader's convenience, in Appendix A.

The main difficulty faced in this approach consists in setting up, and working with, the right pivot space which allows to get weak solutions, i.e. solutions verifying the energy identity, when both $\alpha$ and $\beta$ are allowed to vanish, identically or in a subset of positive measure. Other approaches are possible, as for example the use of a contraction argument, but our approach has the advantage to set up a working framework useful for further studies of the problem.

The first outcome of it is given by the following regularity result, which proof constitutes the second aim of the paper. Before stating it we introduce the exponents $l = l(m, \mu, N)$ and $\lambda = \lambda(m, \mu, N)$ by

$$l = \min\left\{2, \tfrac{\max\{m,r_\Omega\}}{m-1}, \tfrac{\max\{\mu,r_\Gamma\}}{\mu-1}\right\}, \; \lambda = \begin{cases} \infty & \text{if } m \le r_\Omega, \mu \le r_\Gamma, \\ \min\{m', \mu'\} & \text{otherwise.} \end{cases} \tag{1.15}$$

**Theorem 1.3** (**Regularity I**). *Suppose that (I-II) and* (1.8) *hold true, that* $\Omega$ *is* $C^2$ *and* $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. *Then, if*

$$(u_0, u_1) \in W^{2,l} \times H^{1,m,\mu}_{\alpha,\beta}, \tag{1.16}$$

$$-\Delta u_0 + \alpha P_0(u_1) \in L^2(\Omega), \quad \partial_\nu u_{0|\Gamma_1} - \Delta_\Gamma u_{0|\Gamma_1} + \beta Q_0(u_{1|\Gamma}) \in L^2(\Gamma_1), \tag{1.17}$$

*then the weak maximal solution* $u$ *of problem* (1.2) *found in Theorem 1.1 enjoys the further regularity*

$$u \in L^\lambda([0, T_{max}); W^{2,l}) \cap C^1_w([0, T_{max}); H^1) \cap W^{2,\infty}_{loc}([0, T_{max}); H^0), \tag{1.18}$$

$$u' \in C_w([0, T_{max}); H^{1,m,\mu}_{\alpha,\beta}). \tag{1.19}$$

*Moreover,* $u_{tt} - \Delta u + \alpha P_0(u_t) = f_0(u)$ *in* $L^l(\Omega)$, *a.e. in* $(0, T_{max})$, *and* $(u_{|\Gamma})_{tt} + \partial_\nu u - \Delta_\Gamma u_{|\Gamma} + \beta Q_0((u_{|\Gamma})_t) = g_0(u_{|\Gamma})$ *in* $L^l(\Gamma_1)$, *a.e. in* $(0, T_{max})$.

*If* $(u_0, u_1) \in [W^{2,l} \cap H^{1,m,\mu}_{\alpha,\beta}] \times H^{1,m,\mu}_{\alpha,\beta}$ *and* (1.17) *holds, then* (1.18) *becomes*

$$u \in L^\lambda([0, T_{max}); W^{2,l} \cap H^{1,m,\mu}_{\alpha,\beta}) \cap C^1_w([0, T_{max}); H^{1,m,\mu}_{\alpha,\beta}) \cap W^{2,\infty}_{loc}([0, T_{max}); H^0).$$

The regularity (1.17) is improved, depending on the growth of $P_0$, $Q_0$, as follows.

**Theorem 1.4** (**Regularity II**). *Suppose that (I-II) and* (1.8) *hold true, that* $\Omega$ *is* $C^2$ *and* $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. *Moreover suppose that*

$$1 < m \le r_\Omega, \qquad \text{and} \quad 1 < \mu \le r_\Gamma. \tag{1.20}$$

*Then for initial data satisfying* (1.16)–(1.17) *the weak maximal solution* $u$ *of problem* (1.2) *found in Theorem 1.1 enjoys the regularity*

$$u \in C_w([0, T_{max}); W^{2,l}) \cap C^1_w([0, T_{max}); H^1) \cap C^2_w([0, T_{max}); H^0). \tag{1.21}$$

*In particular, when*

$$1 < m \le 1 + r_\Omega/2, \qquad \text{and} \quad 1 < \mu \le 1 + r_\Gamma/2, \tag{1.22}$$

*for initial data* $(u_0,u_1)\in H^2\times H^1$ *we have the optimal regularity*

$$u\in C_w([0,T_{max});H^2)\cap C^1_w([0,T_{max});H^1)\cap C^2_w([0,T_{max});H^0). \tag{1.23}$$

*Remark* 1.2. In the particular case (1.22) Theorem 1.4 sharply extends [66, Theorem 1], dealing with the case $\alpha\equiv 0$, $P_0=f_0\equiv 0$, $\beta\equiv 1$, $Q_0(v)=|v|^{\mu-2}v$, $g_0(u)=|u|^{q-2}u$.

The main difficulty in the proof of Theorems 1.3–1.4 consists in getting the regularity with respect to the space variable on $\Gamma_1$ expressed by (1.21), especially when (1.20) fails to hold and $\Omega$ is merely $C^2$.

The third aim of the paper is to show that, under suitable assumptions on the nonlinearities involved beside (I–II) and (1.8), the semi–flow generated by problem (1.2) is a dynamical system in the phase space $H^1\times H^0$ and, when also (III) holds, in $H^{1,\rho,\theta}_{\alpha,\beta}\times H^0$ for $(\rho,\theta)$ verifying (1.13). By Theorems 1.1–1.2 these assertions hold true if and only if $T_{\max}=\infty$ for all $(u_0,u_1)\in H^1\times H^0$.

To motivate the need of additional assumptions we shall preliminarily show that assumptions (I–II) and (1.8) do not guarantee by themselves that all solutions are global in time, since in some cases they blow–up in finite time. To shortly prove this assertion we temporarily restrict to linear damping terms, that is we replace assumption (I) with the following one:

(I)$'$ $P_0(v)=Q_0(v)=v$ for all $v\in\mathbb{R}$.

To state our blow–up result we introduce

$$\mathfrak{F}_0(u)=\int_0^u f_0(s)\,ds,\qquad \mathfrak{G}_0(u)=\int_0^u g_0(s)\,ds\qquad\text{for all } u\in\mathbb{R}, \tag{1.24}$$

and we make the following specific blow–up assumption:

(IV) $(f_0,g_0)\not\equiv 0$ and there are $\overline{p},\overline{q}>2$ such that

$$f_0(u)u\ge \overline{p}\,\mathfrak{F}_0(u)\ge 0\quad\text{and}\quad g_0(u)u\ge \overline{q}\,\mathfrak{G}_0(u)\ge 0\quad\text{for all } u\in\mathbb{R}. \tag{1.25}$$

*Remark* 1.3. Clearly $f_1$ and $g_1$ in (1.3) satisfy (IV) if and only if

$$c_1=c_2=0,\qquad \gamma,\widetilde{\gamma},\delta,\widetilde{\delta}\ge 0,\qquad \gamma+\widetilde{\gamma}+\delta+\widetilde{\delta}>0,\qquad\text{and}\quad \widetilde{p},\widetilde{q}>2. \tag{1.26}$$

We also introduce the energy functional $\mathcal{E}_0\in C^1(H^1\times H^0)$ defined by

$$\mathcal{E}_0(u_0,u_1)=\frac12\|u_1\|^2_{H^0}+\tfrac12\int_\Omega|\nabla u_0|^2+\tfrac12\int_{\Gamma_1}|\nabla_\Gamma u_0|^2_\Gamma-\int_\Omega\mathfrak{F}_0(u_0)-\int_{\Gamma_1}\mathfrak{G}_0(u_0). \tag{1.27}$$

**Theorem 1.5** **(Blow–up).** *Let (I)$'$, (II), (IV) and* (1.8) *hold. Then*

(i) $N_0:=\{(u_0,u_1)\in H^1\times H^0:\ \mathcal{E}_0(u_0,u_1)<0\}\neq\emptyset$, *and*
(ii) *for any* $(u_0,u_1)\in N_0$ *the unique maximal weak solution* $u$ *of* (1.1) *blows–up in finite time, that is* $T_{\max}<\infty$, *and*

$$\lim_{t\to T^-_{max}}\|u(t)\|_{H^1}+\|u'(t)\|_{H^0}=\lim_{t\to T^-_{max}}\|u(t)\|_p^p+\|u(t)\|^q_{q,\Gamma_1}=\infty. \tag{1.28}$$

*Remark* 1.4. When $(f_0,g_0)\equiv 0$ the set $N_0$ is trivially empty, and all solutions are global in time, as it will be clear from Theorem 1.6. The two cases $f_0\not\equiv 0$, $g_0\equiv 0$ and $f_0\equiv 0$, $g_0\not\equiv 0$, are of particular interest, since they show that just one source, internal or at the boundary, forces solutions to blow–up.

The proof of Theorem 1.5 is based on Theorem 1.1 and on the classical concavity method of H. Levine. In this way we give a first application of Theorem 1.1.

Theorem 1.5 implicitly suggests that all solutions of (1.2) can be global in time when $f_0$ and $g_0$ are sinks, that is $f_0(u)u, g_0(u)u \le 0$ in $\mathbb{R}$, or they are sources, that is $f_0(u)u, g_0(u)u \ge 0$ in $\mathbb{R}$, with at most linear growth at infinity. It is reasonable to extend this conjecture to sums of terms of these types. Moreover nonlocalized damping terms, whose growths at infinity dominate those of the sources (when sources are superlinear), may also prevent solutions to blow–up in finite time. To treat all these cases in a unified framework we shall make, beside (I–II) and (1.8), the following specific global existence assumption:

(V) there are $p_1$ and $q_1$ satisfying

$$2 \le p_1 \le \min\{p, \max\{2, m\}\} \quad \text{and} \quad 2 \le q_1 \le \min\{q, \max\{2, \mu\}\} \tag{1.29}$$

and such that

$$\overline{\lim_{|u|\to\infty}} \mathfrak{F}_0(u)/|u|^{p_1} < \infty \quad \text{and} \quad \overline{\lim_{|u|\to\infty}} \mathfrak{G}_0(u)/|u|^{q_1} < \infty. \tag{1.30}$$

We also suppose that

$$\operatorname{ess\,inf}_\Omega > 0 \text{ if } p_1 > 2 \text{ and } \operatorname{ess\,inf}_{\Gamma_1} \beta > 0 \text{ if } q_1 > 2. \tag{1.31}$$

Since $\mathfrak{F}_0(u) = \int_0^1 f_0(su)u\,ds$ (and similarly $\mathfrak{G}_0$), (V) is a weak version [4] of the following assumption, which is adequate for most purposes and easier to verify:

(V)$'$ there are $p_1$ and $q_1$ such that (1.29) holds with (1.31) and

$$\overline{\lim_{|u|\to\infty}} f_0(u)u/|u|^{p_1} < \infty \qquad \text{and} \quad \overline{\lim_{|u|\to\infty}} g_0(u)u/|u|^{q_1} < \infty. \tag{1.32}$$

*Remark* 1.5. Assumptions (II) and (V)$'$ hold true when $f_0$ (respectively $g_0$) belongs to one among the following classes:

(0) $f_0$ (respectively $g_0$) is constant;
(1) $f_0$ (respectively $g_0$) satisfies (II) with $p \le \max\{2, m\}$ and $\operatorname{ess\,inf}_\Omega \alpha > 0$ if $p > 2$ (respectively $q \le \max\{2, \mu\}$ and $\operatorname{ess\,inf}_{\Gamma_1} \beta > 0$ if $q > 2$);
(2) $f_0$ (respectively $g_0$) satisfies (II) and it is a sink.

More generally (II) and (V)$'$ hold when

$$f_0 = f_0^0 + f_0^1 + f_0^2, \qquad \text{and} \quad g_0 = g_0^0 + g_0^1 + g_0^2, \tag{1.33}$$

where $f_0^i$ and $g_0^i$ are of class (i) for $i = 0, 1, 2$. [5]

*Remark* 1.6. One easily checks that $f_1$ in (1.3) satisfies (II) and (V) if and only if one among the following cases (the analogous cases apply to $g_1$) occurs:

(i) $\gamma > 0$, $p \le \max\{2, m\}$ and $\operatorname{ess\,inf}_\Omega \alpha > 0$ if $p > 2$;
(ii) $\gamma \le 0$, $\widetilde{\gamma} > 0$, $\widetilde{p} \le \max\{2, m\}$ and $\operatorname{ess\,inf}_\Omega \alpha > 0$ if $\widetilde{p} > 2$;
(iii) $\gamma, \widetilde{\gamma} \le 0$.

---

[4] Actually (V)$'$ is more general than (V). Indeed, when $f_0(u) = (m+1)|u|^{m-1}u \cos |u|^{m+1}$ and $g_0(u) = (\mu+1)|u|^{\mu-1}u \cos |u|^{\mu+1}$, (1.32) holds only for $p_1 \ge m+1$, $q_1 \ge \mu+1$, while (1.30) does with $p_1 = q_1 = 2$.

[5] Actually *all* functions verifying (II) and (V)$'$ are of the form (1.33), where $f_0^1$ are $g_0^1$ are sources. See Remark 6.3.

Our global existence result is the following one.

**Theorem 1.6** (**Global existence).** *Let (I–II), (V) and* (1.8) *hold. Then for any* $(u_0, u_1) \in H^1 \times H^0$ *the unique maximal weak solution* $u$ *of* (1.1) *is global in time, that is* $T_{\max} = \infty$. *Consequently the semi–flow generated by problem* (1.2) *is a dynamical system in* $H^1 \times H^0$ *and, when also (III) holds, in* $H^{1,\rho,\theta}_{\alpha,\beta} \times H^0$ *for* $(\rho, \theta)$ *verifying* (1.13).

*Remark* 1.7. Comparing Remarks 1.3 and 1.6 it is clear that, when $P_1$ and $Q_1$ are linear (hence $m = \mu = 2$), $f_1$ and $g_1$ in (1.3) cannot satisfy (IV) *and* (V). By integrating (1.25) one easily sees that the same fact holds for $f_0$ and $g_0$, so the assumptions sets of Theorems 1.5 and 1.6 have empty intersection, as expected. Moreover, since when $m = \mu = 2$ and $\widetilde{\gamma} = \widetilde{\delta} = c_1 = c_2 = 0$ then (V) holds if and only if $p = 2$ when $\gamma > 0$ and $q = 2$ when $\delta > 0$, comparing with (1.26) and remembering that $\widetilde{p} \le p$ and $\widetilde{q} \le q$, we see that Theorem 1.6 is sharp when damping terms are linear and sources are pure powers. The author is convinced that Theorem 1.6 is sharp also when sources are not pure powers and damping terms are nonlinear. He intends to study this topic in a forthcoming paper.

The proof of Theorem 1.6 relies on Theorems 1.1 and 1.2 (so we are giving another application of them), on a suitable auxiliary functional inspired by [28] and on suitable estimates.

The paper is organized as follows: in Section 2 we introduce some background material, we set up the functional setting used and we prove a couple of preliminary results, one of which concerning weak solutions for a linear version of (1.1). In Section 3 we state our main local well–posedness result for (1.1) and a slightly more general (and abstract) version of it, which contains Theorems 1.1–1.2 as particular cases. They are proved in Section 4. Section 5 is devoted to regularity results for problem (1.1) and the proofs of Theorems 1.3–1.4. In Section 6 we give our blow–up and global existence results for problem (1.1) and the proofs of Theorems 1.5–1.6. Finally, Appendix A deals with abstract Cauchy problems for locally Lipschitz perturbations of maximal monotone operators, while in Appendix B we give the proof of the isomorphism property of the operator $-\Delta_M + I$ on a $C^2$ manifold $M$.

## 2. **Background and preliminary results**

2.1. **Notation.** We shall adopt the standard notation for functions spaces in $\Omega$ such as the Lebesgue and Sobolev spaces of integer order, for which we refer to [1]. All the function spaces considered in the paper will be spaces of real valued functions, but for Appendix B where complex–valued functions will be considered. Given a Banach space $X$ and its dual $X'$ we shall denote by $\langle \cdot, \cdot \rangle_X$ the duality product between them. Finally, we shall use the standard notation for vector valued Lebesgue and Sobolev spaces in a real interval.

Given $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma)$, $\alpha, \beta \ge 0$ and $\rho \in [1, \infty]$ we shall respectively denote by $(L^\rho(\Omega), \|\cdot\|_\rho)$, $(L^\rho(\Gamma), \|\cdot\|_{\rho,\Gamma})$, $(L^\rho(\Gamma_1), \|\cdot\|_{\rho,\Gamma_1})$, $(L^\rho(\Omega; \lambda_\alpha), \|\cdot\|_{\rho,\alpha})$, $(L^\rho(\Gamma; \lambda_\beta), \|\cdot\|_{\rho,\beta,\Gamma})$ and $(L^\rho(\Gamma_1; \lambda_\beta), \|\cdot\|_{\rho,\beta,\Gamma_1})$ the Lebesgue spaces (and norms) with respect to the following measures: the standard Lebesgue one in $\Omega$, the hypersurface measure $\sigma$ on $\Gamma$ and $\Gamma_1$, $\lambda_\alpha$ in $\Omega$ defined (for Lebesgue measurable sets) by $d\lambda_\alpha = \lambda_\alpha \, dx$, $\lambda_\beta$ on $\Gamma$ and $\Gamma_1$ defined (for $\sigma$ measurable sets) by $d\lambda_\beta = \lambda_\beta \, d\sigma$. The equivalence

classes with respect to the measures $\lambda_\alpha$ and $\lambda_\beta$ will be respectively denoted by $[\cdot]_\alpha$ and $[\cdot]_\beta$. As usual $\rho'$ is the Hölder conjugate of $\rho$, i.e. $1/\rho + 1/\rho' = 1$.

Finally $W^{\tau,\rho}(\Omega)$, $\tau \geq 0$ will denotes, when $\tau \notin \mathbb{N}_0$, the fractional Sobolev (Sobolev–Slobodeckij) space in $\Omega$ and $H^\tau(\Omega) = W^{\tau,2}(\Omega)$. See [1, 33] or [59].

2.2. **Sobolev spaces and Riemannian gradient on $\Gamma$.** The Sobolev spaces on $\Gamma$ are treated in many textbooks when $\Gamma$ is the boundary of a smooth open bounded set $\Omega \subset \mathbb{R}^N$ or more generally a smooth compact manifold, the non–optimality of the smoothness assumption being often asserted. See for example [35, 41, 42, 59, 60]. An exception is given by [33], so referring to it, when $\Gamma = \partial\Omega$ and $\Omega$ is $C^k$, $k \in \mathbb{N}$, $\Gamma'$ is a relatively open subset of $\Gamma$, $\rho \in (1,\infty)$ and $\theta \in [-k,k]$, we shall denote by $W^{\theta,\rho}(\Gamma')$ ($H^\theta = W^{\theta,2}$) the space defined through local charts, making the reader aware that distributions in $\Gamma'$ are elements of $[C^k_c(\Gamma')]'$. One sees by elementary considerations that this approach is equivalent to the one used in [41] in the smooth case, with local charts and partitions of the unity, and moreover both extend to $C^k$ compact manifolds.

We also recall (see [33, Theorem 1.5.1.2 and 1.5.1.3 p. 37]) the trace operator $u \mapsto u_{|\Gamma}$ which is linear and bounded from $W^{1,\rho}(\Omega)$ to $W^{1-\frac{1}{\rho},\rho}(\Gamma)$, and has a right inverse $\mathbb{D} \in \mathcal{L}(W^{1-\frac{1}{\rho},\rho}(\Gamma), W^{1,\rho}(\Omega))$, i.e. $(\mathbb{D}u)_{|\Gamma} = u$ for all $u$, independently on $\rho$.

We recall here, for the reader's convenience, some well–known preliminaries on the Riemannian gradient, where only the fact that $\Gamma$ is a $C^1$ compact manifold endowed with a $C^0$ Riemannian metric is used. In Appendix B these preliminaries will be used for abstract compact manifolds $M$. We refer to [57] for more details and proofs, given there for smooth manifolds, and to [54] for a general background on differential geometry on $C^k$ manifolds when $k \in \mathbb{N}$.

We start by fixing some notation. We denote by $(\cdot,\cdot)_\Gamma$ the metric inherited from $\mathbb{R}^N$, with $|\cdot|^2_\Gamma = (\cdot,\cdot)_\Gamma$, given in local coordinates $(y_1,\dots,y_{N-1})$ by $(g_{ij})_{i,j=1,\dots,N-1}$. We denote by $d\sigma$ the natural volume element on $\Gamma$, given by $\sqrt{\widetilde{g}}\, dy_1 \wedge \ldots \wedge dy_{N-1}$, where $\widetilde{g} = \det(g_{ij})$. We denote by $(\cdot|\cdot)_\Gamma$ the Riemannian (real) inner product on 1-forms on $\Gamma$ associated to the metric, given in local coordinates by $(g^{ij}) = (g_{ij})^{-1}$. Trivially, as $\Gamma$ is compact, there are $c_i = c_i(\Gamma) > 0$, $i = 1,2,3$, such that [6]

$$c_1 \leq \widetilde{g} \leq c_2, \qquad \text{and} \quad g^{ij}\xi_i\xi_j \geq c_3|\xi|^2 \qquad \text{on } \Gamma, \text{ for all } \xi \in \mathbb{R}^{N-1}. \tag{2.1}$$

We also denote by $d_\Gamma$ the total differential on $\Gamma$ and by $\nabla_\Gamma$ the Riemannian gradient, given in local coordinates by $\nabla_\Gamma u = g^{ij}\,\partial_j u\,\partial_i$ for any $u \in H^1(\Gamma)$. It is then clear that $(d_\Gamma u|d_\Gamma v)_\Gamma = (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma$ for $u, v \in H^1(\Gamma)$, so the use of vectors or forms in the sequel is optional. It is well–known (see [57] in the smooth setting, and the recent paper [36] in the $C^1$ setting) that $H^1(\Gamma)$ can be equipped with the inner product and norm, equivalent to the standard one, given by

$$(u,v)_{H^1(\Gamma)} = \int_\Gamma uv\, d\sigma + \int_\Gamma (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma\, d\sigma, \quad \|u\|^2_{H^1(\Gamma)} = \|u\|^2_{2,\Gamma} + \|\nabla_\Gamma u\|^2_{2,\Gamma} \tag{2.2}$$

for $u, v \in H^1(\Gamma)$, where $\|\nabla_\Gamma u\|^2_{2,\Gamma} := \int_\Gamma |\nabla_\Gamma u|^2_\Gamma$.

[6] here and in the sequel the summation convention is used

In the sequel, the notation $d\sigma$ will be dropped from the boundary integrals; we hope that the reader will be able to put in the appropriate integration elements in all formulas.

2.3. **Functional setting.** We start by giving some details on $L^{2,\rho}_\alpha(\Omega)$ and $L^{2,\rho}_\beta(\Gamma_1)$, which definition can be extended to $\rho=\infty$ by setting, for any $\rho\in[1,\infty]$,

$$L^{2,\rho}_\alpha(\Omega)=\{u\in L^2(\Omega):[u]_\alpha\in L^\rho(\Omega,\lambda_\alpha)\},\quad \|\cdot\|_{2,\rho,\alpha}=\|\cdot\|_2+\|[\cdot]_\alpha\|_{\rho,\alpha},$$
$$L^{2,\rho}_\beta(\Gamma_1)=\{u\in L^2(\Gamma_1):[u]_\beta\in L^\rho(\Gamma_1,\lambda_\beta)\}, \|\cdot\|_{2,\rho,\beta}=\|\cdot\|_{2,\Gamma_1}+\|[\cdot]_\beta\|_{\rho,\beta,\Gamma_1}.$$

Since the case $1\le\rho<2$ reduces to $\rho=2$, we shall consider $2\le\rho\le\infty$ in the sequel. As $u\mapsto(u,[u]_\alpha)$ isometrically embeds $L^{2,\rho}(\Omega)$ into $L^2(\Omega)\times L^\rho(\Omega,\lambda_\alpha)$ (the same argument applying to $L^{2,\rho}(\Gamma_1)$), they are reflexive spaces provided $\rho<\infty$.

Moreover we have the trivial embeddings $L^\rho(\Omega)\hookrightarrow L^{2,\rho}_\alpha(\Omega)$ and $L^\rho(\Gamma_1)\hookrightarrow L^{2,\rho}_\alpha(\Gamma_1)$, which are dense by [51, Theorem 1.17, p. 15] and Lebesgue Dominated Convergence Theorem in abstract measure spaces. For the same reason $[\cdot]_\alpha\in\mathcal{L}(L^{2,\rho}_\alpha(\Omega),L^\rho(\Omega,\lambda_\alpha))$ and $[\cdot]_\beta\in\mathcal{L}(L^{2,\rho}_\beta(\Gamma_1),L^\rho(\Gamma_1,\lambda_\beta))$ have dense ranges. Hence by [16, Corollary 2.18, p.45] their Banach adjoints are injective. Consequently, making the standard identifications

$$[L^\rho(\Omega)]'\simeq L^{\rho'}(\Omega),\qquad\text{and}\quad [L^\rho(\Gamma_1)]'\simeq L^{\rho'}(\Gamma_1), \tag{2.3}$$

when $\rho\in[2,\infty)$ we have the two chains of embeddings [7]

$$[L^\rho(\Omega,\lambda_\alpha)]'\hookrightarrow[L^{2,\rho}_\alpha(\Omega)]'\hookrightarrow L^{\rho'}(\Omega),\ [L^\rho(\Gamma_1,\lambda_\beta)]'\hookrightarrow[L^{2,\rho}_\beta(\Gamma_1)]'\hookrightarrow L^{\rho'}(\Gamma_1). \tag{2.4}$$

Next, given $\rho,\theta\in[2,\infty)$ and $-\infty\le a<b\le\infty$ we introduce the Banach space

$$L^{2,\rho,\theta}_{\alpha,\beta}(a,b)=L^\rho(a,b\,;L^{2,\rho}_\alpha(\Omega))\times L^\theta(a,b\,;L^{2,\theta}_\beta(\Gamma_1)) \tag{2.5}$$

with the standard norm of the product. Clearly it is reflexive and

$$[L^{2,\rho,\theta}_{\alpha,\beta}(a,b)]'\simeq L^{\rho'}(a,b\,;[L^{2,\rho}_\alpha(\Omega))]'\times L^{\theta'}(a,b\,;[L^{2,\theta}_\beta(\Gamma_1)]'). \tag{2.6}$$

Consequently, by (2.4)–(2.6) we have the embedding

$$L^{\rho'}(a,b\,;[L^\rho(\Omega,\lambda_\alpha)]')\times L^{\theta'}(a,b\,;[L^\theta(\Gamma_1,\lambda_\beta)]')\hookrightarrow[L^{2,\rho,\theta}_{\alpha,\beta}(a,b)]'. \tag{2.7}$$

We now give some details on $H^0$ and $H^1$ introduced in Section 1. The standard scalar product of $H^0$ will be denoted by $(\cdot,\cdot)_{H^0}$, The space $H^1$ introduced in (1.5) will be endowed with a scalar product which induces a norm equivalent to the one inherited by the product. We recall [62, Lemma 1] that the space

$$H^1(\Omega;\Gamma)=\{(u,v)\in H^1(\Omega)\times H^1(\Gamma):v=u_{|\Gamma}\}$$

can be equipped with the scalar product [8]

$$(u,v)_{H^1(\Omega;\Gamma)}=\int_\Omega\nabla u\nabla v+\int_\Gamma(\nabla_\Gamma u,\nabla_\Gamma v)_\Gamma+\int_\Gamma uv,\qquad u,v\in H^1(\Omega;\Gamma). \tag{2.8}$$

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[7] $[L^\rho(\Omega,\lambda_\alpha)]'$ and $[L^\rho(\Gamma_1,\lambda_\beta)]'$ cannot be identified with $L^{\rho'}(\Omega,\lambda_\alpha)$ and $L^{\rho'}(\Gamma_1,\lambda_\beta)$, since these identifications would be incoherent with (2.3)

[8] the proof does not depends on the $C^\infty$ regularity of $\Omega$ asserted there

Since $H^1$ is actually a closed subspace of $H^1(\Omega;\Gamma)$, $\nabla_\Gamma u = 0$ a.e. on the relative interior of $\Gamma_0$, that is on $\Gamma\setminus\overline{\Gamma_1}$, and $\sigma(\overline{\Gamma}_0\cap\overline{\Gamma}_1)=0$, we can equip $H^1$ with the scalar product

$$(u,v)_{H^1}=\int_\Omega \nabla u\nabla v+\int_{\Gamma_1}(\nabla_\Gamma u,\nabla_\Gamma v)_\Gamma+\int_{\Gamma_1}uv,\qquad u,v\in H^1. \tag{2.9}$$

The related norm will be denoted by $\|\cdot\|_{H^1}$. Finally the definition of $H^{1,\rho,\theta}_{\alpha,\beta}$ given in (1.6) can be extended also when $\rho=\infty$ and $\theta=\infty$, and it is a reflexive space provided $\rho,\theta\in[2,\infty)$.

The relations among the spaces introduced above when $\rho,\theta\in[2,\infty]$, are given by the following two chains of trivial embeddings

$$H^{1,\rho,\theta}_{\alpha,\beta}\hookrightarrow H^1\hookrightarrow H^0,\qquad\text{and}\qquad H^{1,\rho,\theta}_{\alpha,\beta}\hookrightarrow L^{2,\rho}_\alpha(\Omega)\times L^{2,\theta}_\beta(\Gamma_1)\hookrightarrow H^0. \tag{2.10}$$

At a first glance they are all trivially dense when $\rho,\theta\in[2,\infty)$ since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$ and in $L^2(\Omega)$, while $C^1(\Gamma)$ is dense in $H^1(\Gamma)$ and in $L^2(\Gamma)$. A more careful check of this argument would convince the reader that, even in the simpler case $\Gamma_0=\emptyset$, the density of $C^\infty(\overline{\Omega})$ in $H^1(\Omega)$ and of $C^1(\Gamma)$ in $H^1(\Gamma)$ *does not trivially implies* the density of $\{(u,v)\in C^1(\overline{\Omega})\times C^1(\Gamma): v=u_{|\Gamma}\}$ in $H^1(\Omega;\Gamma)$. See also [36, Remark 2.6] and [40, Lecture 11]. For this reason we give the following result.

**Lemma 2.1.** *Let $\alpha\in L^\infty(\Omega)$, $\beta\in L^\infty(\Gamma_1)$, $\alpha,\beta\ge 0$, and $\rho,\theta\in[2,\infty)$. Then $H^{1,\infty,\infty}_{1,1}$ is dense in both $L^{2,\rho}_\alpha(\Omega)\times L^{2,\theta}_\beta(\Gamma_1)$ and $H^{1,\rho,\theta}_{\alpha,\beta}$. Then all the embeddings in* (2.10) *are dense.*

*Proof.* We first claim that $H^{1,\infty,\infty}_{1,1}$ is dense in $L^{2,\rho}_\alpha(\Omega)\times L^{2,\theta}_\beta(\Gamma_1)$. By (2.4) our claim follows once we prove that, given $(\varphi,\psi)\in L^{\rho'}(\Omega)\times L^{\theta'}(\Gamma_1)$ such that

$$\int_\Omega\varphi u+\int_{\Gamma_1}\psi u=0\qquad\text{for all } u\in H^{1,\infty,\infty}_{1,1}, \tag{2.11}$$

then $(\varphi,\psi)=0$. Taking $u\in C^\infty_c(\Omega)$ in (2.11) we immediately get that $\varphi=0$, hence (2.11) reduces to $\int_{\Gamma_1}\psi u=0$ for all $u\in H^{1,\infty,\infty}_{1,1}$. Next, given any $v\in C^1_c(\Gamma\setminus\overline{\Gamma_1})$, trivially extended to $v\in C^1(\Gamma)$, we have $v\in H^1(\Gamma)\cap W^{1-\frac{1}{2N},2N}(\Gamma)$, hence $\mathbb{D}v\in W^{1,2N}(\Omega)$, so $\mathbb{D}v\in H^1(\Omega)\cap C(\overline{\Omega})$ by Morrey's theorem, hence $\mathbb{D}v\in H^{1,\infty,\infty}_{1,1}$. It follows that $\int_{\Gamma_1}\psi v=0$ for all $v\in C^1_c(\Gamma\setminus\overline{\Gamma_1})$, so $\psi=0$ a.e. in $\Gamma\setminus\overline{\Gamma_1}$, from which, as $\sigma(\overline{\Gamma}_0\cap\overline{\Gamma}_1)=0$, we get $\psi=0$, concluding the proof of our claim.

To prove that $H^{1,\infty,\infty}_{1,1}$ is dense in $H^{1,\rho,\theta}_{\alpha,\beta}$ we use a classical truncation argument. Given $u\in H^1$ and $k\in\mathbb{N}$ we respectively denote by $u^k$ and $(u_{|\Gamma})^k$ the truncated of $u$ and $u_{|\Gamma}$ given by

$$u^k=\begin{cases}u & \text{if } |u|\le k,\\ ku/|u| & \text{if } |u|>k,\end{cases}\qquad (u_{|\Gamma})^k=\begin{cases}u_{|\Gamma} & \text{if } |u_{|\Gamma}|\le k,\\ ku_{|\Gamma}/|u_{|\Gamma}| & \text{if } |u_{|\Gamma}|>k,\end{cases} \tag{2.12}$$

or $u^k=k-[2k-(u+k)^+]^+$ and $(u_{|\Gamma})^k=k-[2k-(u_{|\Gamma}+k)^+]^+$. Trivially $u^k\in L^\infty(\Omega)$ and $(u_{|\Gamma})^k\in L^\infty(\Gamma)$. Moreover using [30, Lemma 7.6] first in $\Omega$ and then in coordinate neighborhoods on $\Gamma$, we get that $u^k\in H^1(\Omega)$, $(u_{|\Gamma})^k\in H^1(\Gamma)$,

$$\nabla u^k=\begin{cases}\nabla u & \text{if } |u|\le k,\\ 0 & \text{if } |u|>k,\end{cases}\qquad\text{and}\qquad \nabla_\Gamma(u_{|\Gamma})^k=\begin{cases}\nabla_\Gamma u_{|\Gamma} & \text{if } |u_{|\Gamma}|\le k,\\ 0 & \text{if } |u_{|\Gamma}|>k.\end{cases} \tag{2.13}$$

Now we note that

$$(u_{|\Gamma})^k = u^k{}_{|\Gamma}, \tag{2.14}$$

which is trivial when $u \in H^1 \cap C(\overline{\Omega})$, while in the general case $u \in H^1$ it follows by approximating $u$ by a sequence $(u_n)_n$ in $C^\infty(\overline{\Omega})$ such that $u_n \to u$ in $H^1(\Omega)$ (see [16, Corollary 9.8, p. 277]). By (2.14) then we have $u^k \in H^{1,\infty,\infty}_{1,1}$ for all $k \in \mathbb{N}$. We now note that, by (2.12)–(2.13) and several applications of the Lebesgue Dominated Convergence Theorem, we get $u^k \to u$ in $H^{1,\rho,\theta}_{\alpha,\beta}$ as $k \to \infty$. Hence $H^{1,\infty,\infty}_{1,1}$ is dense in $H^{1,\rho,\theta}_{\alpha,\beta}$.

Finally we note that the density of the embeddings in (2.10) follows from previous statements since $H^{1,\infty,\infty}_{1,1} \subseteq H^{1,\rho,\theta}_{\alpha,\beta}$, $H^1 = H^{1,2,2}_{1,1}$ and $H^0 = L^{2,2}_1(\Omega) \times L^{2,2}_1(\Gamma_1)$. □

Using (2.10) and Lemma 2.1 and making the identification $(H^0)' \simeq H^0$, which is coherent with (2.3), we have the two chains of dense embeddings

$$H^{1,\rho,\theta}_{\alpha,\beta} \hookrightarrow H^1 \hookrightarrow H^0 \simeq (H^0)' \hookrightarrow (H^1)' \hookrightarrow (H^{1,\rho,\theta}_{\alpha,\beta})', \tag{2.15}$$

$$H^{1,\rho,\theta}_{\alpha,\beta} \hookrightarrow L^{2,\rho}_\alpha(\Omega) \times L^{2,\theta}_\beta(\Gamma_1) \hookrightarrow H^0 \simeq (H^0)' \hookrightarrow [L^{2,\rho}_\alpha(\Omega)]' \times [L^{2,\theta}_\beta(\Gamma_1)]' \hookrightarrow (H^{1,\rho,\theta}_{\alpha,\beta})'$$

and, by (2.4),

$$[L^\rho(\Omega,\lambda_\alpha)]' \times [L^\rho(\Gamma_1,\lambda_\beta)]' \hookrightarrow [L^{2,\rho}_\alpha(\Omega)]' \times [L^{2,\rho}_\beta(\Gamma_1)]' \hookrightarrow (H^{1,\rho,\theta}_{\alpha,\beta})'. \tag{2.16}$$

2.4. **Weak solutions for the linear version of problem (1.1).** We now consider the linear evolution boundary value problem

$$\begin{cases} u_{tt} - \Delta u = \xi & \text{in } (0,T)\times\Omega,\\ u = 0 & \text{on } (0,T)\times\Gamma_0,\\ u_{tt} + \partial_\nu u - \Delta_\Gamma u = \eta & \text{on } (0,T)\times\Gamma_1, \end{cases} \tag{2.17}$$

where $0 < T < \infty$ and $\xi = \xi(t,x)$, $\eta = \eta(t,x)$ are given forcing terms of the form

$$\begin{cases} \xi = \xi_1 + \alpha\xi_2, & \xi_1 \in L^1(0,T;L^2(\Omega)), \quad \xi_2 \in L^{\rho'}(0,T;L^{\rho'}(\Omega,\lambda_\alpha)),\\ \eta = \eta_1 + \beta\eta_2, & \eta_1 \in L^1(0,T;L^2(\Gamma_1)), \quad \eta_2 \in L^{\theta'}(0,T;L^{\theta'}(\Gamma_1,\lambda_\beta)), \end{cases} \tag{2.18}$$

where $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma_1)$, $\alpha,\beta \ge 0$ and $\rho,\theta \in [2,\infty)$.

To write (2.17) in a more abstract form we set $A \in \mathcal{L}(H^1,(H^1)')$ by

$$\langle Au, v\rangle_{H^1} = \int_\Omega \nabla u \nabla v + \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma, \qquad \text{for all } u,v \in H^1. \tag{2.19}$$

Moreover we denote $\Xi_1 = (\xi_1,\eta_1) \in L^1(0,T;H^0)$ and we define $\Xi_2 \in [L^{2,\rho,\theta}_{\alpha,\beta}(0,T)]'$ by setting $\langle \Xi_2(t), \Phi\rangle_{L^{2,\rho}_\alpha(\Omega)\times L^{2,\theta}_\beta(\Gamma_1)} = \int_\Omega \alpha\xi_2(t)\phi + \int_{\Gamma_1} \beta\eta_2(t)\psi$ for almost all $t \in (0,T)$ and all $\Phi = (\phi,\psi) \in L^{2,\rho}_\alpha(\Omega) \times L^{2,\theta}_\beta(\Gamma_1)$. By (2.15) we set $\Xi = \Xi_1 + \Xi_2 \in L^1(0,T;[L^{2,\rho}_\alpha(\Omega)]' \times [L^{2,\theta}_\beta(\Gamma_1)]')$.

The following result characterizes solutions of $u$ in the sense of distributions.

**Lemma 2.2.** *Suppose that* (2.18) *holds and let*

$$u \in L^\infty(0,T;H^1) \cap W^{1,\infty}(0,T;H^0), \qquad u' \in L^{2,\rho,\theta}_{\alpha,\beta}(0,T). \tag{2.20}$$

*Then the following facts are equivalent:*

(i) *the distribution identity*

$$\int_0^T \left[-(u',\phi')_{H^0} + \int_\Omega \nabla u \nabla \phi + \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma \phi)_\Gamma - \int_\Omega \xi\phi - \int_{\Gamma_1} \eta\phi\right] = 0 \tag{2.21}$$

*holds for all* $\phi \in C_c((0,T);H^1) \cap C^1_c((0,T);H^0) \cap L^{2,\rho,\theta}_{\alpha,\beta}(0,T)$*;*

(ii) $u' \in W^{1,1}(0,T;[H^{1,\rho,\theta}_{\alpha,\beta}]')$ *and*

$$u''(t) + Au(t) = \Xi(t) \qquad \text{in } [H^{1,\rho,\theta}_{\alpha,\beta}]' \text{ for almost all } t \in (0,T); \tag{2.22}$$

(iii) *the alternative distribution identity*

$$(u',\phi)_{H^0}\Big|_0^T + \int_0^T \left[-(u',\phi')_{H^0} + \langle Au,\phi\rangle_{H^1} - \langle \Xi,\phi\rangle_{L^{2,\rho}_\alpha(\Omega)\times L^{2,\theta}_\beta(\Gamma_1)}\right] = 0 \tag{2.23}$$

*holds for all* $\phi \in C([0,T];H^1) \cap C^1([0,T];H^0) \cap L^{2,\rho,\theta}_{\alpha,\beta}(0,T)$.

*Moreover any* $u$ *satisfying* (2.20) *and (ii) enjoys the further regularity*

$$u \in C([0,T];H^1) \cap C^1([0,T];H^0) \tag{2.24}$$

*and satisfies the energy identity*

$$\frac12 \|u'\|^2_{H^0} + \frac12 \langle Au,u\rangle_{H^1}\Big|_s^t = \int_s^t \langle \Xi, u'\rangle_{L^{2,\rho}_\alpha(\Omega)\times L^{2,\theta}_\beta(\Gamma_1)}\, d\tau \tag{2.25}$$

*for* $0 \le s \le t \le T$.

*Proof.* Let us denote $\widetilde{X} = H^{1,\rho,\theta}_{\alpha,\beta}$. We first claim that (2.24)–(2.25) hold for any u satisfying (ii). To prove our claim we apply [55, Theorems 4.1 and 4.2]. Referring to the notation of the quoted paper (but adding a $\widetilde{\ }$ on the notation of the spaces) we set

$$\widetilde{V} = H^1, \quad \widetilde{H} = H^0, \quad \widetilde{W} = L^{2,\rho}_\alpha(\Omega) \times L^{2,\theta}_\beta(\Gamma_1), \quad \widetilde{Z} = L^{2,\rho,\theta}_{\alpha,\beta}(0,T),$$

while $A(t) = A$ is defined by (2.19). To check the structural assumptions of [55, p.545] we note that $\widetilde{V}$ and $\widetilde{W}$ are both contained in $\widetilde{H}$ and $\widetilde{X} = \widetilde{V} \cap \widetilde{W}$ is dense in both $\widetilde{V}$ and $\widetilde{W}$ by Lemma 2.1. Moreover, by (2.5)–(2.6) we have $\widetilde{Z} \subset L^1(0,T;\widetilde{W})$ and $\widetilde{Z}' \subset L^1(0,T;\widetilde{W}')$ as dense subsets with continuous inclusions. Trivially for any $w \in \widetilde{Z}$ and $v \in \widetilde{Z}'$ we have $\langle v,w,\rangle_{\widetilde{Z}} = \int_0^T \langle v(t),w(t)\rangle_{\widetilde{W}}\, dt$ so [55, (3.1)] holds. Next multiplications by step functions trivially maps $\widetilde{Z}$ into itself and translations in $t$ are continuous in the strong operator topology of $\widetilde{Z}$ thanks to the extension of [16, Lemma 4.3, p.114] for vector–valued functions. The specific assumptions for [55, Theorems 4.1 and 4.2] are satisfied since $\widetilde{V}$ is dense in $\widetilde{H}$ by Lemma 2.1, $\widetilde{W}$ is contained in $\widetilde{H}$ and [55, (3.5)] holds by (2.9) and (2.19). Since [55, (4.1)] in this case reduces to (2.22), the proof of our first claim is completed.

Next we claim that (i) and (ii) are equivalent each other and with the distribution identity

$$\langle u',\phi\rangle_X\Big|_0^T + \int_0^T \left[-(u',\phi')_{H^0} + \langle Au,\phi\rangle_{H^1} - \langle \Xi,\phi\rangle_{L^{2,\rho}_\alpha(\Omega)\times L^{2,\theta}_\beta(\Gamma_1)}\right] = 0 \tag{2.26}$$

for all $\phi \in C^1([0,T];\widetilde{X})$. Indeed if $u$ satisfies (i) then by taking test functions $\phi$ in the separate form $\phi(t) = \psi(t)w$, $\psi \in C^\infty_c(0,T)$, $w \in \widetilde{X}$, from (2.21) we immediately

get that $\int_0^T -u\psi' + (Au - \Xi)\psi = 0$ in $\widetilde{X}'$ , from which (ii) follows. Conversely, if (ii) holds then, by a standard density argument in $W^{1,1}(0,T;\widetilde{X}')$ we get that for any $\phi \in C^1([0,T];\widetilde{X})$ we have $\langle u', \phi\rangle_X \in W^{1,1}(0,T)$ and

$$\langle u', \phi\rangle_{\widetilde{X}'} = \langle u'', \phi\rangle_{\widetilde{X}} + \langle u', \phi'\rangle_{\widetilde{X}} \qquad \text{a.e. in } (0,T). \tag{2.27}$$

Then, evaluating (2.22) with $\phi \in C^1([0,T];\widetilde{X})$, integrating in $[0,T]$ and using (2.27) we get (2.26). By (2.26) we immediately get that that (2.21) holds true for any $\phi \in C^1_c((0,T);\widetilde{X})$ and then, by standard time regularization, for any $\phi \in C_c((0,T);H^1) \cap C^1_c((0,T);H^0) \cap L^{2,\rho,\theta}_{\alpha,\beta}(0,T)$, so completing the proof of our claim.

Since (iii) trivially implies (i), the proof is completed (thanks to our second claim) provided we prove that if (2.26) holds for all $\phi \in C^1([0,T];\widetilde{X})$ then (2.23) holds for all $\phi \in C([0,T];H^1) \cap C^1([0,T];H^0) \cap L^{2,\rho,\theta}_{\alpha,\beta}(0,T)$. Since by (2.24) the identity (2.26) can be rewritten as (2.23), we just have to prove that we can take less regular test functions in it. By standard time regularization one easily get that (2.23) holds for any $\phi \in C(\mathbb{R};H^1) \cap C^1(\mathbb{R};H^0) \cap L^{2,\rho,\theta}_{\alpha,\beta}(-\infty,\infty)$, so our claim follows since any $\phi \in C([0,T];H^1) \cap C^1([0,T];H^0) \cap L^{2,\rho,\theta}_{\alpha,\beta}(0,T)$ can be extended to $\widetilde{\phi} \in C(\mathbb{R};H^1) \cap C^1(\mathbb{R};H^0) \cap L^{2,\rho,\theta}_{\alpha,\beta}(-\infty,\infty)$ [9] . □

## 3. Well–posedness in $H^1 \times H^0$ and in $H^{1,\rho,\theta}_{\alpha,\beta} \times H^0$: statements

In this section we state our main well–posedness result for problem (1.1) and a slightly more general (and abstract) version of it. With reference to (1.1) we now introduce our main assumptions on the nonlinearities in it, starting from $P$ and $Q$.

(PQ1) $P$ and $Q$ are Carathéodory functions, respectively in $\Omega \times \mathbb{R}$ and $\Gamma_1 \times \mathbb{R}$, and there are $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma_1)$, $\alpha, \beta \geq 0$, and constants $m, \mu > 1$, $c_m, c_\mu > 0$ such that

$$|P(x,v)| \leq c_m \alpha(x)(1 + |v|^{m-1}) \text{ for almost all } x \in \Omega, \text{ all } v \in \mathbb{R}; \tag{3.1}$$

$$|Q(x,v)| \leq c_\mu \beta(x)(1 + |v|^{\mu-1}) \text{ for almost all } x \in \Gamma_1, \text{ all } v \in \mathbb{R}. \tag{3.2}$$

(PQ2) $P$ and $Q$ are monotone increasing in the second variable for almost all values of the first one;

(PQ3) $P$ and $Q$ are coercive, that is there are constants $c'_m, c'_\mu > 0$ such that

$$P(x,v)v \geq c'_m \alpha(x)|v|^m \qquad \text{for almost all } x \in \Omega, \text{ all } v \in \mathbb{R}; \tag{3.3}$$

$$Q(x,v)v \geq c'_\mu \beta(x)|v|^\mu \qquad \text{for almost all } x \in \Gamma_1, \text{ all } v \in \mathbb{R}. \tag{3.4}$$

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[9] One first defines $\phi_1 \in C([-T/2,3T/2];H^1) \cap C^1([-T/2,3T/2];H^0) \cap L^{2,\rho,\theta}_{\alpha,\beta}(-T/2,3T/2)$ as

$$\phi_1(t) = \begin{cases} \phi(t) & \text{if } t \in [0,T], \\ 3\phi(-t) - 2\phi(-2t) & \text{if } t \in [-T/2,0], \\ 3\phi(2T-t) - 2\phi(3T-2t) & \text{if } t \in [T,3T/2], \end{cases} .$$

Then one sets $\widetilde{\phi} \in C_c((-T/2,3T/2);H^1) \cap C^1_c((-T/2,3T/2);H^0) \cap L^{2,\rho,\theta}_{\alpha,\beta}(-T/2,3T/2)$ as $\tilde{\phi} = \psi_0\phi_1$ where $\psi_0 \in C^\infty_c(-T/2,3T/2)$ and $\psi_0 = 1$ in $[0,T]$.

*Remark* 3.1. Trivially (PQ1–3) yield $P(\cdot,0)\equiv 0$ and $Q(\cdot,0)\equiv 0$. Moreover in the separate variable case considered in problem (1.2), that is $P(x,v)=\alpha(x)P_0(v)$ and $Q(x,v)=\beta(x)Q_0(v)$ with $\alpha\in L^\infty(\Omega)$, $\beta\in L^\infty(\Gamma_1)$, $\alpha,\beta\geq 0$, (PQ1–3) reduce to assumption (I).

Referring to (PQ1) we fix the notation

$$\overline{m}=\max\{2,m\},\ \overline{\mu}=\max\{2,\mu,\},\ W=L^{2,\overline{m}}_\alpha(\Omega)\times L^{2,\overline{\mu}}_\beta(\Gamma_1),\ X=H^{1,\overline{m},\overline{\mu}}_{\alpha,\beta}, \tag{3.5}$$

so (2.15) and the subsequent embedding read as

$$\begin{aligned} X &\hookrightarrow H^1 \hookrightarrow H^0 \simeq (H^0)' \hookrightarrow (H^1)' \hookrightarrow X',\\ X &\hookrightarrow W \hookrightarrow H^0 \simeq (H^0)' \hookrightarrow W' \hookrightarrow X'. \end{aligned} \tag{3.6}$$

Moreover, for $-\infty\leq a<b\leq\infty$, we denote

$$Z(a,b)=L^{2,\overline{m},\overline{\mu}}_{\alpha,\beta}(a,b),\qquad \text{and}\quad Z'(a,b)=[Z(a,b)]'.$$

By (PQ1) the Nemitskii operators $\widehat{P}$ and $\widehat{Q}$ (respectively) associated to $P$ and $Q$ are continuous from $L^{\overline{m}}(\Omega)$ to $L^{\overline{m}'}(\Omega))\simeq[L^{\overline{m}}(\Omega))]'$ and from $L^{\overline{\mu}}(\Gamma_1)$ to $L^{\overline{\mu}'}(\Gamma_1))\simeq[L^{\overline{\mu}}(\Gamma_1))]'$. By (PQ1) they can be uniquely extended to

$$\widehat{P}:L^{2,\overline{m}}_\alpha(\Omega)\to[L^{\overline{m}}(\Omega,\lambda_\alpha)]'\quad\text{and}\quad \widehat{Q}:L^{2,\overline{\mu}}_\beta(\Gamma_1)\to[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]' \tag{3.7}$$

given, for $u\in L^{2,\overline{m}}_\alpha(\Omega)$, $v\in L^{\overline{m}}(\Omega,\lambda_\alpha)$, $w\in L^{2,\overline{\mu}}_\beta(\Gamma_1)$ and $z\in L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)$, by

$$\langle\widehat{P}(u),v\rangle_{L^{\overline{m}}(\Omega,\lambda_\alpha)}=\int_\Omega P(\cdot,u)v\quad\text{and}\quad\langle\widehat{Q}(w),z\rangle_{L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)}=\int_{\Gamma_1}Q(\cdot,w)z. \tag{3.8}$$

We denote

$$B=(\widehat{P},\widehat{Q}):W\to[L^{\overline{m}}(\Omega,\lambda_\alpha)]'\times[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]', \tag{3.9}$$

and we point out some relevant properties of $B$ we shall use in the sequel.

**Lemma 3.1.** *Let (PQ1—2) hold and $(a,b)\subset\mathbb{R}$ is bounded. Then*

(i) *$B$ is continuous and bounded from $W$ to $[L^{\overline{m}}(\Omega,\lambda_\alpha)]'\times[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]'$ and hence, by* (2.16)*, to $W'$;*
(ii) *$B$ acts boundedly and continuously from $Z(a,b)$ to $L^{\overline{m}'}(a,b\,;[L^{\overline{m}}(\Omega,\lambda_\alpha)]')\times L^{\overline{\mu}'}(a,b\,;[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]')$ and hence, by* (2.7)*, to $Z'(a,b)$;*
(iii) *$B$ is monotone in $W$ and in $Z(a,b)$.*

*Proof.* We shall prove the properties listed above only for $\widehat{P}$, since the same arguments apply, *mutatis mutandis*, to $\widehat{Q}$. As to (i) we note that the fact that $\widehat{P}$ is well–defined and bounded follows from (3.1) and Hölder inequality. Moreover, since the classical result on the continuity of Nemitskii operators (see [3, Theorem 2.2, p.16]) trivially extends to abstract measure spaces, the Nemitskii operator associated to $P_\alpha=P/\alpha$ (which is defined $\lambda_\alpha$ – a.e. in $\Omega$) is continuous from $L^{\overline{m}}(\Omega,\lambda_\alpha)$ to $L^{\overline{m}'}(\Omega,\lambda_\alpha)$. By the form of the Riesz isomorphism between $L^{\overline{m}'}(\Omega,\lambda_\alpha)$ and $[L^{\overline{m}}(\Omega,\lambda_\alpha)]'$, since $[\cdot]_\alpha\in\mathcal{L}(L^{2,\overline{m}}_\alpha(\Omega),L^{\overline{m}}(\Omega,\lambda_\alpha))$, we get (i). To prove (ii) we note that the boundedness of $B$, almost everywhere defined in $(a,b)$ by (3.9), is a trivial consequence of (PQ1) and Hölder inequality once again. To prove the asserted continuity we note that, by repeating previous argument, the Nemitskii operator $\widehat{P_\alpha}$ associated to $P_\alpha=P/\alpha$ is continuous from $L^{\overline{m}}((a,b)\times\Omega,\lambda'_\alpha)$ ($\lambda'_\alpha$ denoting the product of the 1 - dimensional Lebesgue measure and $\lambda_\alpha$) to $L^{\overline{m}'}((a,b)\times\Omega,\lambda'_\alpha)$.

Since for any $\rho \in [1,\infty)$ one can prove as in the standard case, by the density of $C_c((a,b)\times\Omega)$ in $L^{\overline{m}'}((a,b)\times\Omega,\lambda'_\alpha)$ (cfr. [51, Theorem 1.36 p. 27 and Theorem 3.14 p. 68]), that

$$L^\rho((a,b)\times\Omega,\lambda'_\alpha)\simeq L^\rho(a,b;L^\rho(\Omega,\lambda_\alpha)), \tag{3.10}$$

we then get that $\widehat{P}$ is continuous from $L^{\overline{m}}(a,b;L^{\overline{m}}(\Omega,\lambda_\alpha))$ to $L^{\overline{m}}(a,b;[L^{\overline{m}}(\Omega,\lambda_\alpha)]')$ and then by (2.4) we get (ii). Finally (iii) trivially follows from (PQ2). □

We introduce the following assumption, which will be used only in the last part of the proof of Theorem 3.2 below:

(PQ4) if $m > r_\Omega$ there are constants $c''_m, M_m > 0$ such that

$$P_v(x,v)\ge c''_m\,\alpha(x)|v|^{m-2}\quad \text{for almost all } (x,v)\in\Omega\times(\mathbb{R}\setminus(-M_m,M_m)), \tag{3.11}$$

and if $\mu > r_\Gamma$ there are constants $c''_\mu, M_\mu > 0$ such that

$$Q_v(x,v)\ge c''_\mu\,\beta(x)|v|^{\mu-2}\quad \text{for almost all } (x,v)\in\Gamma_1\times(\mathbb{R}\setminus(-M_\mu,M_\mu)). \tag{3.12}$$

*Remark* 3.2. Since by (PQ1–2) the partial derivatives $P_v$ and $Q_v$ exist almost everywhere (see [22]) and are nonnegative, (3.11)–(3.12) always hold if one allows $c''_m$ and $c''_\mu$ to vanish, and the assumption (PQ4) reduces to ask that if $m > r_\Omega$ then there is $M_m > 0$ such that one can take $c''_m > 0$ in (3.11) and if $\mu > r_\Gamma$ then there is $M_\mu > 0$ such that one can take $c''_\mu > 0$ in (3.12). Moreover, in the separate variable case considered in problem (1.2), that is $P(x,v)=\alpha(x)P_0(v)$ and $Q(x,v)=\beta(x)Q_0(v)$ with $\alpha\in L^\infty(\Omega)$, $\beta\in L^\infty(\Gamma_1)$, $\alpha,\beta\ge 0$, (PQ4) reduces to (III).

*Remark* 3.3. We remark here, for a future use, some trivial consequences of assumptions (PQ1–4). Setting $c'_m=0$ when $m\le r_\Omega$ and $c'_\mu=0$ when $\mu\le r_\Gamma$, since $P_v,Q_v\ge 0$ a.e., from (PQ4) we have

$$P_v(x,v)\ge\alpha(x)\left[c''_m|v|^{m-2}-c'''_m\right]\quad \text{for almost all } (x,v)\in\Omega\times\mathbb{R}, \tag{3.13}$$

$$Q_v(x,v)\ge\beta(x)\left[c''_\mu|v|^{\mu-2}-c'''_\mu\right]\quad \text{for almost all } (x,v)\in\Gamma_1\times\mathbb{R}, \tag{3.14}$$

where $c'''_m=c''_mM_m^{m-2}$, $c'''_\mu=c''_\mu M_\mu^{\mu-2}$. By (PQ2) then, integrating (3.13) we get, for almost all $x\in\Omega$ and all $v<w$,

$$P(x,w)-P(x,v)\ge\alpha(x)\left[\frac{c''_m}{m-1}\left(|w|^{m-2}w-|v|^{m-2}v\right)-c'''_m(w-v)\right]. \tag{3.15}$$

Consequently, using when $m > r_\Omega$ the elementary inequality

$$\left(|w|^{m-2}w-|v|^{m-2}v\right)(w-v)\ge\widetilde{c_m}|w-v|^m\qquad\text{for all } v,w\in\mathbb{R},$$

where $\widetilde{c_m}$ is a positive constant, setting $\widetilde{c_m}''=c''_m\widetilde{c_m}/(m-1)$, from (3.15) we get

$$\widetilde{c_m}''\alpha(x)|v-w|^{\overline{m}}\le c'''_m\alpha(x)|v-w|^2+(P(x,w)-P(x,v))(w-v) \tag{3.16}$$

for almost all $x\in\Omega$ and all $v,w\in\mathbb{R}$, with $\widetilde{c_m}''>0$ when $m > r_\Omega$. Using the same arguments we get the existence of $\widetilde{c_\mu}''\ge 0$ such that

$$\widetilde{c_\mu}''\beta(x)|v-w|^{\overline{\mu}}\le c'''_\mu\beta(x)|v-w|^2+(Q(x,w)-Q(x,v))(w-v) \tag{3.17}$$

for almost all $x\in\Gamma_1$ and all $v,w\in\mathbb{R}$, with $\widetilde{c_\mu}''>0$ when $\mu > r_\Gamma$.

Our main assumptions on $f$ and $g$, are the following ones:

(FG1) $f$ and $g$ are Carathéodory functions, respectively in $\Omega\times\mathbb{R}$ and $\Gamma_1\times\mathbb{R}$, and there are constants $p,q\geq 2$ and $c_p,c_q\geq 0$ such that

$$|f(x,u)|\leq c_p(1+|u|^{p-1}),\qquad \text{for almost all } x\in\Omega, \text{ all } u\in\mathbb{R}, \text{ and} \tag{3.18}$$

$$|g(x,u)|\leq c_q(1+|u|^{q-1})\qquad \text{for almost all } x\in\Gamma_1, \text{ all } u\in\mathbb{R}; \tag{3.19}$$

(FG2) there are constants $c_p',c_q'\geq 0$ such that

$$|f(x,u)-f(x,v)|\leq c_p'|u-v|(1+|u|^{p-2}+|v|^{p-2}) \tag{3.20}$$

for almost all $x\in\Omega$, all $u,v\in\mathbb{R}$, and

$$|g(x,u)-g(x,v)|\leq c_q'|u-v|(1+|u|^{q-2}+|v|^{q-2}) \tag{3.21}$$

for almost all $x\in\Gamma_1$, all $u,v\in\mathbb{R}$.

*Remark* 3.4. Assumptions (FG1–2) can be equivalently formulated as follows:

(FG1)$'$ $f$ and $g$ are Carathéodory functions, respectively in $\Omega\times\mathbb{R}$ and $\Gamma_1\times\mathbb{R}$, $f(x,\cdot)\in C^{0,1}_{\text{loc}}(\mathbb{R})$ for almost all $x\in\Omega$ and $g(x,\cdot)\in C^{0,1}_{\text{loc}}(\mathbb{R})$ for almost all $x\in\Gamma_1$;

(FG2)$'$ $f(\cdot,0)\in L^\infty(\Omega)$ and $g(\cdot,0)\in L^\infty(\Gamma_1)$;

(FG3)$'$ there are constants $p,q\geq 2$, $\widetilde{c_p},\widetilde{c_q}\geq 0$ such that

$$|f_u(x,u)|\leq \widetilde{c_p}(1+|u|^{p-2}),\qquad \text{for almost all } (x,u)\in\Omega\times\mathbb{R}, \text{ and}$$

$$|g_u(x,u)|\leq \widetilde{c_q}(1+|v|^{q-2})\qquad \text{for almost all } (x,u)\in\Gamma_1\times\mathbb{R}.$$

Indeed by (FG1–2) we immediately get (FG1–2)$'$. Moreover, by (FG1)$'$ $f_u$ and $g_u$ exist almost everywhere [10] so (FG3)$'$ follows. Conversely one gets (FG1–2) by simply integrating (FG3)$'$ with respect to the second variable in the convenient interval.

In the case considered in problem (1.2), i.e. $f(x,u)=f_0(u)$ and $g(x,u)=g_0(u)$, assumptions (FG1–2) then reduce to (II). Other relevant examples of functions $f$ and $g$ satisfying (FG1–2) are given by

$$\begin{aligned} f_2(x,u)&=\gamma_1(x)|u|^{\widetilde{p}-2}u+\gamma_2(x)|u|^{p-2}u+\gamma_3(x),\quad 2\leq\widetilde{p}\leq p,\ \gamma_i\in L^\infty(\Omega),\\ g_2(x,u)&=\delta_1(x)|u|^{\widetilde{q}-2}u+\delta_2(x)|u|^{q-2}u+\delta_3(x),\quad 2\leq\widetilde{q}\leq q,\ \delta_i\in L^\infty(\Gamma_1),\end{aligned} \tag{3.22}$$

and by

$$f_3(x,u)=\gamma(x)f_0(u),\quad g_3(x,u)=\delta(x)g_0(u),\quad \gamma\in L^\infty(\Omega),\quad \delta\in L^\infty(\Gamma_1), \tag{3.23}$$

where $f_0$ and $g_0$ satisfy (II).

In line with Sobolev embedding of $H^1(\Omega)$ the source $f$ can be classified (see [15]) as subcritical (or critical) when $2\leq p\leq 1+r_\Omega/2$, supercritical when $1+r_\Omega/2<p\leq r_\Omega$ and supersupercritical when $p>r_\Omega$. The source $g$ can be classified in the same way referring to $r_\Gamma$. This paper is devoted to the case when both sources are subcritical

[10] the fact that measurable functions in an open set, which are locally absolutely continuous with respect to a variable, possess almost everywhere partial derivative with respect to that variable is classical, as stated for example in [43, p.297]. However the sceptical reader can prove it by repeating [22, Proof of Proposition 2.1 p. 173] for Carathéodory functions, so getting the measurability of the four Dini derivatives. Hence the set where the derivative does not exist is measurable and finally it has zero measure by Fubini's theorem.

(or critical), that is (1.8) holds. In this case is easy to see, using Hölder inequality and Sobolev embedding, that the Nemitskii operators $\widehat{f} : H^1(\Omega) \to L^2(\Omega)$ and $\widehat{g} : H^1(\Gamma) \cap L^2(\Gamma_1) \to L^2(\Gamma_1)$ respectively associated to $f$ and $g$ are locally Lipschitz.

To precise the meaning of weak solutions of problem (1.1) we first note that, by (FG1–2) , for any $u$ satisfying (2.20), denoting $u' = (u_t, (u_{|\Gamma})_t)$, we have $\widehat{f}(u) \in L^1(0,T; L^2(\Omega))$ and $\widehat{g}(u_{|\Gamma}) \in L^1(0,T; L^2(\Gamma_1))$. Moreover, by Lemma 3.1, $\widehat{P}(u_t) \in L^{\overline{m}'}(0,T; [L^{\overline{m}}(\Omega,\lambda_\alpha)]')$ and $\widehat{Q}(u_{|\Gamma})_t) \in L^{\overline{\mu}'}(0,T; [L^{\overline{\mu}}(\Gamma_1,\lambda_\alpha)]')$. Then, by (3.10), $\widehat{P}(u_t) = \alpha\, \xi_2$ and $\widehat{Q}(u_{|\Gamma})_t) = \beta\, \eta_2$, with $\xi_2 \in L^{\overline{m}'}(0,T; L^{\overline{m}'}(\Omega,\lambda_\alpha))$ and $\eta_2 \in L^{\overline{\mu}'}(0,T; L^{\overline{\mu}'}(\Gamma_1,\lambda_\beta))$. By previous remarks and Lemma 2.2 the following definition makes sense.

**Definition 3.1.** Let (PQ1–3), (FG1–2) hold and $u_0 \in H^1$, $u_1 \in H^0$. A weak solution of problem (1.1) in $[0,T]$, $0 < T < \infty$, is $u$ verifying (2.20)– (2.21) with

$$\xi = \widehat{f}(u) - \widehat{P}(u_t), \quad \eta = \widehat{g}(u_{|\Gamma}) - \widehat{Q}((u_{|\Gamma})_t), \quad \rho = \overline{m} \quad \text{and} \quad \theta = \overline{\mu}, \tag{3.24}$$

such that $u(0) = u_0$ and $u'(0) = u_1$. A weak solution of (1.1) in $[0,T)$, $0 < T \le \infty$, is $u \in L^\infty_{\text{loc}}([0,T); H^1)$ which is a weak solution of (1.1) in $[0,T']$ for any $T' \in (0,T)$. Such a solution is called maximal if it has no proper extensions.

*Remark* 3.5. The introduction of Definition 3.1 is justified by the fact that, when $\Gamma$ is $C^2$, any $u \in C^2([0,T] \times \overline{\Omega})$ is a weak solution of (1.1) if and only if it is a classical one.

*Remark* 3.6. It follows by Lemma 2.2 that any weak solution of (1.1) in dom $u = [0,T]$ or dom $u = [0,T)$ enjoys the further regularity

$$u \in C(\text{dom } u; H^1) \cap C^1(\text{dom } u; H^0), \tag{3.25}$$

satisfies the energy identity

$$\tfrac{1}{2}\left[\int_\Omega u_t^2 + \int_{\Gamma_1} (u_{|\Gamma})_t^2 + \int_\Omega |\nabla u|^2 + \int_{\Gamma_1} |\nabla_\Gamma u|_\Gamma^2\right]_s^t + \int_s^t \int_\Omega P(\cdot,u_t)u_t \\ + \int_s^t \left[\int_{\Gamma_1} Q(\cdot,(u_{|\Gamma})_t)(u_{|\Gamma})_t - \int_\Omega f(\cdot,u)u_t - \int_{\Gamma_1} g(\cdot,u)(u_{|\Gamma})_t\right] = 0 \tag{3.26}$$

for all $s,t \in$ dom $u$, and the distribution identity

$$\left[\int_\Omega u_t\phi + \int_{\Gamma_1} (u_{|\Gamma})_t\phi\right]_0^{T'} + \int_0^{T'} \left[-\int_\Omega u_t\phi_t - \int_{\Gamma_1} (u_{|\Gamma})_t(\phi_{|\Gamma})_t + \int_\Omega \nabla u \nabla\phi \right.\\ \left. + \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma \phi)_\Gamma + \int_\Omega P(\cdot,u_t)\phi + \int_{\Gamma_1} Q(\cdot,(u_{|\Gamma})_t)\phi - \int_\Omega f(\cdot,u)\phi - \int_{\Gamma_1} g(\cdot,u)\phi\right] = 0$$

for all $T' \in$ dom $u$ and $\phi \in C([0,T']; H^1) \cap C^1([0,T']; H^0) \cap Z(0,T')$.

Finally we remark that when $u_0 \in H^{1,\rho,\theta}_{\alpha,\beta}$ for some finite $\rho$, $\theta$ satisfying (1.13) then, as $u' \in L^1(0,T'; H^{1,\rho,\theta}_{\alpha,\beta})$ for all $T' \in$ dom $u$, one easily gets that $u \in W^{1,1}(0,T'; H^{1,\rho,\theta}_{\alpha,\beta})$, so

$$u \in C(\text{dom } u; H^{1,\rho,\theta}_{\alpha,\beta}). \tag{3.27}$$

We can now state our main local well–posedness result for problem (1.1).

**Theorem 3.1.** *Let (PQ1–3), (FG1–2),* (1.8) *hold,* $u_0 \in H^1$ *and* $u_1 \in H^0$*. Then problem* (1.1) *has a unique maximal weak solution* $u = u(u_0, u_1)$ *in* $[0, T_{max})$*,* $T_{max} = T_{max}(u_0, u_1) > 0$*. Moreover*

$$\lim_{t \to T_{max}^-} \|u(t)\|_{H^1} + \|u'(t)\|_{H^0} = \infty$$

*provided* $T_{max} < \infty$*. Next, if* $(u_{0n}, u_{1n}) \to (u_0, u_1)$ *in* $H^1 \times H^0$*, denoting* $u_n = u(u_{0n}, u_{1n})$ *and* $T^n_{max} = T_{max}(u_{0n}, u_{1n})$*, we have*

(i) $T_{max} \le \liminf_n T^n_{max}$*, and*
(ii) $u_n \to u$ *in* $C([0, T^*]; H^1) \cap C^1([0, T^*]; H^0)$ *for all* $T^* \in (0, T_{max})$*.*

*Finally, if also (PQ4) holds and* $(u_{0n}, u_{1n}) \to (u_0, u_1)$ *in* $H^1 \times H^0$ *we also have* $u'_n \to u'$ *in* $Z(0, T^*)$ *for all* $T^* \in (0, T_{max})$ *and consequently, if* $(u_{0n}, u_{1n}) \to (u_0, u_1)$ *in* $H^{1,\rho,\theta}_{\alpha,\beta} \times H^0$ *for some* $\rho, \theta$ *satisfying* (1.13)*, we also have* $u_n \to u$ *in* $C([0, T^*]; H^{1,\rho,\theta}_{\alpha,\beta})$ *for all* $T^* \in (0, T_{max})$*.*

Theorem 3.1 is a particular case of an analogous result concerning a slightly more general and abstract version of problem (1.1), that is the abstract Cauchy problem

$$\begin{cases} u'' + Au + B(u') = F(u) & \text{in } X', \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{3.28}$$

where $A$ and $B$ are the operators respectively defined in (2.19) and (3.9), and $F : H^1 \to H^0$ is a locally Lipschitz map, that is for any $R > 0$ there is $L(R) \ge 0$ such that

$$\|F(u) - F(v)\|_{H^0} \le L(R)\, \|u - v\|_{H^1} \qquad \text{provided } \|u\|_{H^1}, \|v\|_{H^1} \le R. \tag{3.29}$$

When (FG1–2) and (1.8) hold, $F = (\widehat{f}, \widehat{g})$ satisfies (3.29).

We first precise the meaning of strong, generalized and weak solutions of

$$u'' + Au + B(u') = F(u) \qquad \text{in } X'. \tag{3.30}$$

**Definition 3.2.** Let (PQ1–3) and (3.29) hold, and $0 < T < \infty$.

(i) By a *strong* solution of (3.30) in $[0, T]$ we mean $u \in W^{1,\infty}(0, T; H^1) \cap W^{2,\infty}(0, T; H^0)$ such that $Au(t) + B(u'(t)) \in H^0$ and $u'(t) \in X$ for all $t \in [0, T]$ and (3.30) holds in $H^0$ almost everywhere in $(0, T)$.
(ii) By a *generalized* solution of (3.30) in $[0, T]$ we mean the limit of a sequence of strong solutions of (3.30) in $C([0, T]; H^1) \cap C^1([0, T]; H^0)$.
(iii) By a *weak* solution of (3.30) in $[0, T]$ we mean $u$ satisfying (2.20) with $\rho = \overline{m}$, $\theta = \overline{\mu}$ and the distribution identity

$$\int_0^T -(u', \phi')_{H^0} + \langle Au, \phi \rangle_{H^1} + \langle B(u'), \phi \rangle_W = \int_0^T (F(u), \phi)_{H^0} \tag{3.31}$$

for all $\phi \in C_c((0, T); H^1) \cap C^1_c((0, T); H^0) \cap Z(0, T)$.

By a solution in $[0, T)$, $T \in (0, \infty]$, we mean $u \in L^\infty_{\text{loc}}([0, T); H^1)$ which is a solution in $[0, T']$ for any $T' \in (0, T)$. Such a solution is called maximal if has no proper extensions in the same class.

*Remark* 3.7. For any weak solution of (3.30) in $[0,T]$ we have $F(u) \in L^\infty(0,T;H^0)$, hence as in Remark 3.6 we see that weak solutions satisfy (3.25) as well as the generalized versions of the energy and distribution identities in Remark 3.6. Moreover for any couple $(u,v)$ of weak solutions the energy identity

$$\frac{1}{2}\|w'\|_{H^0}^2+\frac{1}{2}\langle Aw,w\rangle_{H^1}\Big|_s^t+\int_s^t\langle B(u')-B(v'),w'\rangle_W=\int_s^t(F(u)-F(v),w')_{H^0} \tag{3.32}$$

holds for $s,t \in \text{dom u} \cap \text{dom v}$, where $w$ denotes the difference $u-v$. Finally also in this case (3.27) holds true for $u_0 \in H^{1,\rho,\theta}_{\alpha,\beta}$, with $(\rho,\theta)$ satisfying (1.13).

By previous remark the following definition makes sense.

**Definition 3.3.** By a strong, generalized or weak solution of (3.28) we mean a solution of (3.30) in the corresponding class verifying also the initial conditions.

Our main result concerning (3.28) is the following one.

**Theorem 3.2.** *Let (PQ1–3),* (3.29) *hold,* $u_0 \in H^1$ *and* $u_1 \in H^0$*. Then problem* (3.28) *has a unique maximal weak solution* $u = u(u_0,u_1)$ *in* $[0,T_{max})$*,* $T_{max} = T_{max}(u_0,u_1) > 0$*, which is also the unique maximal generalized solution of it. If*

$$u_0 \in H^1, \quad u_1 \in X, \quad \text{and} \quad Au_0 + B(u_1) \in H^0, \tag{3.33}$$

*then* $u$ *is actually the unique maximal strong solution of* (3.28)*. Moreover*

$$\lim_{t\to T_{max}^-}\|u(t)\|_{H^1}+\|u'(t)\|_{H^0}=\infty \tag{3.34}$$

*provided* $T_{max} < \infty$*, and* $T_{max} = \infty$ *when* $F$ *is globally Lipschitz.*

*Next, if* $(u_{0n},u_{1n}) \to (u_0,u_1)$ *in* $H^1 \times H^0$*, denoting* $u_n = u(u_{0n},u_{1n})$ *and* $T^n_{max} = T_{max}(u_{0n},u_{1n})$*, we have*

(i) $T_{max} \le \liminf_n T^n_{max}$*, and*
(ii) $u_n \to u$ *in* $C([0,T^*];H^1) \cap C^1([0,T^*];H^0)$ *for all* $T^* \in (0,T_{max})$*.*

*Finally, if also (PQ4) holds and* $(u_{0n},u_{1n}) \to (u_0,u_1)$ *in* $H^1 \times H^0$ *we also have* $u_n' \to u'$ *in* $Z(0,T^*)$ *for all* $T^* \in (0,T_{max})$*. Consequently, if* $(u_{0n},u_{1n}) \to (u_0,u_1)$ *in* $H^{1,\rho,\theta}_{\alpha,\beta} \times H^0$ *for some* $\rho,\theta$ *satisfying* (1.13)*, we also have* $u_n \to u$ *in* $C([0,T^*];H^{1,\rho,\theta}_{\alpha,\beta})$ *for all* $T^* \in (0,T_{max})$*.*

Theorem 3.2 will be proved in the next section by transforming (3.28) in a first order Cauchy problem, applying nonlinear semigroup theory to it, and finally discussing the relations between various type of solutions of (3.28).

## 4. Well–posedness in $H^1 \times H^0$ and in $H^{1,\rho,\theta}_{\alpha,\beta} \times H^0$: proofs

We introduce the phase space for problem (3.28), that is the Hilbert space

$$\mathcal{H} = H^1 \times H^0, \tag{4.1}$$

endowed with the standard scalar product $(\cdot,\cdot)_{\mathcal{H}}$ given by

$$(U_1,U_2)_{\mathcal{H}} = (u_1,u_2)_{H^1} + (v_1,v_2)_{H^0} \qquad \text{for all } U_i = (u_i,v_i),\ i=1,2. \tag{4.2}$$

Moreover, using (3.6), we introduce the nonlinear operator $\mathcal{A}: D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$D(\mathcal{A}) = \{(u,v) \in H^1 \times X : Au + B(v) \in H^0\}, \tag{4.3}$$

$$\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ Au + B(v) \end{pmatrix}, \tag{4.4}$$

and the abstract Cauchy problem

$$\begin{cases} U' + \mathcal{A}U + \mathcal{F}(U) = 0 \qquad \text{in } \mathcal{H}, \\ U(0) = U_0 \in \mathcal{H}, \end{cases} \tag{4.5}$$

where $\mathcal{F}: \mathcal{H} \to \mathcal{H}$ is any locally Lipschitz map.

The meaning of strong and generalized solutions of (4.5) in $[0,T]$, $0 < T < \infty$ is standard (see [53, Theorem 4.1 and Definition, pp.180–183]), while by solutions in $[0,T)$ we mean $U \in C([0,T); \mathcal{H})$ which are solutions in $[0,T']$ in the corresponding sense for all $T' \in (0,T)$. Our main result on problem (4.5) is the following one.

**Theorem 4.1.** *Let (PQ1–3) hold. Then the operator $\mathcal{A}+I$ is maximal monotone in $\mathcal{H}$, $D(\mathcal{A})$ is dense in $\mathcal{H}$ and $A(0)=0$. Consequently, given any locally Lipschitz map $\mathcal{F}: \mathcal{H} \to \mathcal{H}$, the following conclusions hold:*

- (i) *for any $U_0 \in \mathcal{H}$ the problem* (4.5) *has a unique maximal generalized solution $U = U(U_0)$ in $[0, T_{max})$, $T_{max} = T_{max}(U_0)$, which is the unique maximal strong solution of it when $U_0 \in D(\mathcal{A})$;*
- (ii) $\lim_{t \to T_{max}^-} \|U(t)\|_{\mathcal{H}} = \infty$ *provided $T_{max} < \infty$, and $T_{max} = \infty$ provided $\mathcal{F}$ is globally Lipschitz;*
- (iii) *if $U_{0n} \to U_0$ in $\mathcal{H}$ then denoting $U_n = U(U_{0n})$ and $T^n_{max} = T_{max}(U_{0n})$, we have $T_{max} \le \liminf_n T^n_{max}$ and $U_n \to U$ in $C([0,T^*]; \mathcal{H})$ for all $T^* \in (0, T_{max})$.*

*Proof.* **Step 1: $\mathcal{A}+I$ is monotone in $\mathcal{H}$.** Let $U_i = (u_i, v_i) \in D(\mathcal{A})$ for $i = 1,2$. By (4.2) and (4.4)

$$\begin{aligned}(\mathcal{A}(U_1) - \mathcal{A}(U_2), U_1 - U_2)_{\mathcal{H}} \\ = (v_2 - v_1, u_1 - u_2)_{H^1} + (A(u_1 - u_2) + B(v_1) - B(v_2), v_1 - v_2)_{H^0}.\end{aligned} \tag{4.6}$$

Since $v_i \in X$ for $i = 1,2$, by (3.6) we have

$$\begin{aligned}(A(u_1 - u_2) + B(v_1) - B(v_2), v_1 - v_2)_{H^0} \\ = \langle A(u_1 - u_2), v_1 - v_2 \rangle_{H^1} + \langle B(v_1) - B(v_2), v_1 - v_2 \rangle_W.\end{aligned} \tag{4.7}$$

By plugging (2.9), (2.19) and (4.7) in (4.6) we get

$$(\mathcal{A}(U_1) - \mathcal{A}(U_2), U_1 - U_2)_{\mathcal{H}} = \int_{\Gamma_1} (v_2 - v_1)(u_1 - u_2) + \langle B(v_1) - B(v_2), v_1 - v_2 \rangle_W.$$

By Lemma 3.1–(iii), (2.9) and (4.2) we then get

$$\begin{aligned}(\mathcal{A}(U_1) - \mathcal{A}(U_2), U_1 - U_2)_{\mathcal{H}} &\ge \int_{\Gamma_1} (v_2 - v_1)(u_1 - u_2) \\ &\ge -\frac{1}{2}\|v_1 - v_2\|^2_{2,\Gamma_1} - \frac{1}{2}\|u_1 - u_2\|^2_{2,\Gamma_1} \ge -\|U_1 - U_2\|^2_{\mathcal{H}},\end{aligned}$$

and then $\mathcal{A}+I$ is monotone.

**Step 2: $\mathcal{A}+I$ is maximal monotone in $\mathcal{H}$.** By Step 1 and the nonlinear version of Minty's theorem (see [53, Lemma 1.3, p. 159]) this fact is equivalent to prove that $\mathrm{Rg}(\mathcal{A}+2I)=\mathcal{H}$. Consequently, by (4.3)–(4.4) we have to show that for all $(h^0,h^1)\in H^0\times H^1$ the system

$$\begin{cases} 2u-v=h^1 & \text{in } H^1,\\ 2v+Au+B(v)=h^0 & \text{in } X', \end{cases} \tag{4.8}$$

has a solution $(u,v)\in H^1\times X$. Since $X\subset H^1$ we can solve the first equation in $u$ and plug $u=\frac12(v+h^1)$ in the second one. Hence to solve (4.8) reduces to prove that, for $h^2=2h^0-Ah^1\in (H^1)'$, the single equation

$$4v+Av+2B(v)=h^2 \qquad \text{in } X' \tag{4.9}$$

has a solution $v\in X$. Actually we claim that (4.9) has a solution for any $h^2\in X'$ i.e. that the operator $T:X\to X'$ given by $T=4I+A+2B$ is surjective.

We first consider, for the reader's convenience, the simplest linear case when $P(x,v)=\alpha(x)v$ and $Q(x,v)=\beta(x)v$. In this case clearly $m=\overline{m}=\mu=\overline{\mu}=2$, so $X=H^1$ and for all $u,v\in H^1$ we have $\langle T(u),v\rangle_X=a(u,v)$, where $a$ is the continuous bilinear form in $H^1$ given by

$$a(u,v)=4(u,v)_{H^0}+\int_\Omega \nabla u\nabla v+\int_{\Gamma_1}\nabla_\Gamma u\nabla_\Gamma v+\int_\Omega \alpha uv+\int_{\Gamma_1}\beta uv.$$

Since, by (2.9), $a(u,u)\ge \|u\|^2$, $a$ is coercive, so $T$ is surjective by the Lax–Milgram Theorem.

In the general case $T$ is (possibly) nonlinear but, by (2.19) and Lemma 3.1–(iii), it is monotone being the sum of monotone operators. Moreover by Lemma 3.1–(i) and (3.6) we have $T\in C(X,X')$. Next, by (3.5),

$$\begin{aligned}
&\|[u]_\alpha\|_{\overline{m},\alpha}\le \|[u]_\alpha\|_{2,\alpha}+\|[u]_\alpha\|_{m,\alpha}\le \|\alpha\|_\infty\|u\|_2+\|[u]_\alpha\|_{m,\alpha},\\
&\|[v]_\beta\|_{\overline{\mu},\beta}\le \|[v]_\beta\|_{2,\beta,\Gamma_1}+\|[v]_\beta\|_{\mu,\beta,\Gamma_1}\le \|\beta\|_{\infty,\Gamma_1}\|v\|_{2,\Gamma_1}+\|[v]_\beta\|_{\mu,\beta,\Gamma_1}
\end{aligned}$$

for all $u\in L^{2,\overline{m}}_\alpha(\Omega)$ and $v\in L^{2,\overline{\mu}}_\beta(\Gamma_1)$. Consequently, by (1.6) and (3.5), there is $c_1=c_1(\Omega,\|\alpha\|_\infty,\|\beta\|_{\infty,\Gamma_1})>0$ such that

$$\|u\|_X\le c_1(\|u\|_{H^1}+\|[u]_\alpha\|_{m,\alpha}+\|[u]_\beta\|_{\mu,\beta,\Gamma_1})\quad \text{for all } u\in X. \tag{4.10}$$

On the other hand, by (2.9) and (PQ3), for any $u\in X$ we have

$$\begin{aligned}
\langle T(u),u\rangle_X =&4\|u\|^2_{H^0}+\int_\Omega|\nabla u|^2+\int_{\Gamma_1}|\nabla_\Gamma u|^2_\Gamma+2c_m'\int_\Omega P(\cdot,u)u\\
&+2c_\mu'\int_{\Gamma_1}Q(\cdot,u)u\ \ge\ c_2(\|u\|^2_{H^1}+\|[u]_\alpha\|^m_{m,\alpha}+\|[u]_\beta\|^\mu_{\mu,\beta,\Gamma_1})
\end{aligned} \tag{4.11}$$

where $c_2=\min\{4,2c_m'2c_\mu'\}>0$. By the elementary inequality $x^{s'}\le 1+x^s$ for all $0\le s'\le s$, $x\ge 0$, and discrete Hölder inequality, from (4.11) we get

$$\langle T(u),u\rangle_X\ge 3^{1-\mu_0}c_2\left(\|u\|_{H^1}+\|[u]_\alpha\|_{m,\alpha}+\|[u]_\beta\|_{\mu,\beta,\Gamma_1}\right)^{\mu_0}-3c_2 \tag{4.12}$$

where $\mu_0=\min\{2,m,\mu\}$. By combining (4.10) and (4.12), since $\mu_0>1$, we get that $T$ is coercive, i.e. $\|u_n\|_X\to\infty$ implies $\langle T(u_n),u_n\rangle_X/\|u_n\|_X\to\infty$. Then our claim follows since monotone, hemicontinuous and coercive operators are surjective (see [8, Theorem 1.3 p. 40] or [10, Corollary 2.3 p. 37])).[11]

[11] An alternative proof of this point is given in Remark 4.1, page 25.

**Step 3: $\mathcal{A}0 = 0$ and $D(\mathcal{A})$ is dense in $\mathcal{H}$.** The first conclusion follows by (4.4) and Remark 3.1. To prove the second one we note that $H^{1,2(\overline{m}-1),2(\overline{\mu}-1)}_{\alpha,\beta} \subseteq X$ and by (PQ1) we have $B(H^{1,2(\overline{m}-1),2(\overline{\mu}-1)}_{\alpha,\beta}) \subseteq H^0$. Consequently

$$(A+I)^{-1}(H^0) \times H^{1,2(\overline{m}-1),2(\overline{\mu}-1)}_{\alpha,\beta} \subseteq D(\mathcal{A}). \tag{4.13}$$

Now $H^{1,2(\overline{m}-1),2(\overline{\mu}-1)}_{\alpha,\beta}$ is dense in $H^0$ by Lemma 2.1, while $(A+I)^{-1}(H^0)$ is dense in $H^1$ since $H^0$ is dense in $(H^1)'$ by (3.6) and $A+I : H^1 \to (H^1)'$ is an isomorphism by (2.9), (2.19) and Riesz–Fréchet theorem. Hence, by (4.13), $D(\mathcal{A})$ is dense in $\mathcal{H}$.

**Step 4: conclusion.** Assertions (i–iii) follow at once by applying Theorems A.1, A.2 and Remark A.1 in Appendix A to (4.5), which can trivially rewritten as

$$\begin{cases} U' + \mathcal{A}_1 U + \mathcal{F}_1(U) = 0 & \text{in } \mathcal{H}, \\ U(0) = U_0 \in \mathcal{H} \end{cases} \tag{4.14}$$

where $\mathcal{A}_1 = \mathcal{A} + I$ and $\mathcal{F}_1 = \mathcal{F} + I$. □

*Remark* 4.1. The surjectivity of the operator $T$ introduced in **Step 2** also follows by the Direct Method of the Calculus of Variations without invoking the surjectivity theorem of V. Barbu quoted before, since $T$ has a variational nature. Indeed, setting

$$\mathcal{P}(x,v) = \int_0^v P(x,s)\,ds \qquad \text{and} \quad \mathcal{Q}(y,v) = \int_0^v Q(y,s)\,ds$$

for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $v \in \mathbb{R}$, one easily sees that

$$\mathcal{B}(u) = 2\|u\|^2_{H^0} + \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_{\Gamma_1} |\nabla_\Gamma u|^2_\Gamma + \int_\Omega \mathcal{P}(\cdot,u) + \int_{\Gamma_1} \mathcal{Q}(\cdot,u) - \langle h^2, v\rangle_X$$

defines a (possibly nonlinear) functional $\mathcal{B} \in C^1(X)$ and that its Fréchet differential is nothing but $T - h^2$. Moreover, by (4.10) and (3.29), since $\mathcal{B}(0) = 0$, one gets that for all $u \in X$

$$\mathcal{B}(u) = \int_0^1 \frac{d}{dt} J(tu)\,dt = \int_0^1 \langle T(tu) - h^2, u\rangle_X\,dt \ge c_3\|u\|^{\mu_0}_X - \|h^2\|_{X'}\|u\|_X - 3c_2$$

where $c_3 = 3^{1-\mu_0}(\mu_0+1)^{-1}c_2 c_1^{-\mu_0}$. Hence, as $\mu_0 > 1$, $\mathcal{B}$ is coercive in $X$ (that is $\mathcal{B}(u_n) \to \infty$ when $\|u_n\|_X \to \infty$) so $\mathcal{B}$ has a minimum $v \in X$, which is then a critical point of it, so $T(v) = h^2$.

To prove Theorem 3.2 we have to prove that the generalized solution found in Theorem 4.1 is actually a weak solution, which is unique. We start with the uniqueness.

**Lemma 4.1.** *The weak maximal solution of* (3.28) *is unique.*

*Proof.* Clearly the statement reduces to prove that, given two weak solutions $u$ and $v$ in $[0,T]$, $0 < T < \infty$, then $u = v$. We set $M = \max\{\|u\|_{C([0,T];H^1)}, \|v\|_{C([0,T];H^1)}\}$. Using (3.32) and Lemma 3.1 – (iii) and (3.29) we get the estimate

$$\tfrac{1}{2}\|w'\|^2_{H^0} + \tfrac{1}{2}\langle Aw, w\rangle_{H^1}\Big|_0^t \le L(M)\int_0^t \|w\|_{H^1}\|w'\|_{H^0} \qquad \text{for } t \in [0,T]. \tag{4.15}$$

By (2.9) and (2.19) we have $\langle Au,u\rangle_{H^1}=\|u\|_{H^1}^2-\|u\|_{H^0}^2$ for all $u\in H^1$. Moreover, as $w(0)=0$, by Hölder inequality we have

$$\|w(t)\|_{H^0}^2=\left\|\int_0^t w'\right\|_{H^0}^2\le T\int_0^t \|w'\|_{H^0}^2. \tag{4.16}$$

Hence by (4.15) and Young inequality we get

$$\tfrac12\|w'(t)\|_{H^0}^2+\tfrac12\|w(t)\|_{H^1}^2\le \tfrac{T+L(M)}{2}\int_0^t \|w'\|_{H^0}^2+\|w\|_{H^1}^2 \qquad \text{for } t\in[0,T]. \tag{4.17}$$

The proof is completed by applying Gronwall inequality. □

**Lemma 4.2.** *Generalized solutions of* (3.28) *are also weak.*

*Proof.* Clearly we can prove the statement for solutions in $[0,T]$, $0<T<\infty$.

**Step 1: strong solutions are also weak.** Let $u$ be a strong solution. We first claim that $u'\in Z(0,T)$. Since, by Definition 3.2, $u'(t)\in X$ for all $t\in[0,T]$, by (3.30) we get

$$(u'',u')_{H^0}+\langle Au,u'\rangle_{H^1}+\langle B(u'),u'\rangle_W=(F(u),u')_{H^0} \qquad \text{a.e. in } (0,T). \tag{4.18}$$

Since $u\in W^{1,\infty}(0,T;H^1)\cap W^{2,\infty}(0,T;H^0)$, so $Au\in W^{1,\infty}(0,T;(H^1)')$, by standard time–regularization we have $\|u'\|_{H^0}^2, \langle Au,u\rangle_{H^1}\in W^{1,\infty}(0,T)$ and

$$\left(\|u'\|_{H^0}^2\right)'=2(u'',u')_{H^0},\quad \langle Au,u\rangle_{H^1}'=2\langle Au,u'\rangle_{H^1} \qquad \text{a.e. in } (0,T),$$

where the symmetry of $A$ is also used. Since $(F(u),u')_{H^0}\in L^\infty(0,T)$, by (4.18) we then get that $\langle B(u'),u'\rangle_W\in L^\infty(0,T)\subset L^1(0,T)$ as $T<\infty$. Our claim then follows by (PQ3). Combining it with Lemma 3.1–(ii) we get that $B(u')\in L^{\overline{m}'}(0,T\,;[L^{\overline{m}}(\Omega,\lambda_\alpha)]')\times L^{\overline{\mu}'}(0,T\,;[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]')$. Since $F(u)\in L^1(0,T;H^0)$ and trivially $u'\in W^{1,1}(0,T;X')$, by Riesz theorem and Lemma 2.2 we get (3.31), concluding Step 1.

**Step 2: generalized solutions are also weak.** Let $u$ be a generalized solution and $(u_n)_n$ a sequence of strong solutions of (3.30) converging to $u$ in $C([0,T];H^1)\cap C^1([0,T];H^0)$. By Step 1 and Remark 3.7 the energy identity

$$\tfrac12\|u_n'\|_{H^0}^2+\tfrac12\langle Au_n,u_n\rangle_{H^1}\Big|_0^T+\int_0^T\langle B(u_n'),u_n'\rangle_W=\int_0^T(F(u_n),u_n')_{H^0}$$

holds for all $n\in\mathbb{N}$. Since by (3.29) we have $F(u_n)\to F(u)$ in $C([0,T];H^0)$ we can pass to the limit in the last identity to get

$$\tfrac12\|u'\|_{H^0}^2+\tfrac12\langle Au,u\rangle_{H^1}\Big|_0^T+\lim_n\int_0^T\langle B(u_n'),u_n'\rangle_W=\int_0^T(F(u),u')_{H^0}. \tag{4.19}$$

Then, by (PQ3) and Lemma 3.1–(ii), it follows that $u_n'$ and $B(u_n')$ are (respectively) bounded in $Z(0,T)$ and $L^{\overline{m}'}(0,T\,;[L^{\overline{m}}(\Omega,\lambda_\alpha)]')\times L^{\overline{\mu}'}(0,T\,;[L^{\overline{\mu}}(\Gamma_1,\lambda_\beta)]')$. Hence, up to a subsequence, $u_n'\to\psi$ and $B(u_n')\to\chi$ weakly in these spaces. Since $u_n'\to u'$ in $L^2(0,T;H^0)$ and $Z(0,T)\hookrightarrow L^2(0,T;H^0)$, it follows that $\psi=u'$, so $u_n'\to u'$ weakly in $Z(0,T)$. We can pass to the limit in the distribution identity (3.31) written, thanks to Step 1, for $u_n$, and get

$$\int_0^T -(u',\phi')_{H^0}+\langle Au,\phi\rangle_{H^1}+\langle\chi,\phi\rangle_W=\int_0^T(F(u),\phi)_{H^0} \tag{4.20}$$

for all $\phi \in C_c((0,T);H^1) \cap C^1_c((0,T);H^0) \cap Z(0,T)$. By a further application of Lemma 2.2 we then get the energy identity

$$\tfrac12 \|u'\|^2_{H^0} + \tfrac12 \langle Au, u\rangle_{H^1}\Big|_0^T + \int_0^T \langle \chi, u'\rangle_W = \int_0^T (F(u), u')_{H^0}. \tag{4.21}$$

Combining (4.19) and (4.21) we then get $\lim_n \int_0^T \langle B(u_n'), u_n'\rangle_W = \int_0^T \langle \chi, u'\rangle_W$. By Lemma 3.1–(ii–iii) and [8, Theorem 1.3 p.40] $B$ is maximal monotone in $Z(0,T)$, so by the classical monotonicity argument (see [9, Lemma 1.3 p.49] we get $B(u') = \chi$ which, by (4.20), concludes the proof. □

We not turn to the proofs of the results stated in Section 3.

**Proof of Theorem 3.2.** We apply Theorem 4.1 with $\mathcal{F}$ being given by

$$\mathcal{F}(U) = \begin{pmatrix} 0 \\ -F(u) \end{pmatrix}, \qquad \text{where} \quad U = \begin{pmatrix} u \\ v \end{pmatrix},$$

which is trivially locally Lipschitz in $\mathcal{H}$ by (3.29). Consequently for any $(u_0, u_1) \in H^1 \times H^0$ problem (3.28) has a unique maximal generalized solution $u$ in $[0, T_{\max})$, which is the unique strong maximal solution of it when (see (4.3)) also $u_1 \in X$ and $Au_0 + B(u_1) \in H^0$. By Lemmas 4.1 and 4.2 then $u$ is also the unique weak solution of (3.28) in $[0, T_{\max})$. By Theorem 4.1–(ii) we then get (3.34) and the maximality of $u$ among weak solutions of (3.28). The continuous dependence on the data in $H^1 \times H^0$ then follows directly from Theorem 4.1–(iii). Moreover when $F$ is globally Lipschitz also $\mathcal{F}$ is globally Lipschitz, so all solutions are global in time.

To complete the proof we assume from now on that (PQ4) holds. By (3.16)–(3.17) there is $C = C(\|\alpha\|_\infty, \|\beta\|_{\infty,\Gamma_1}, c_m''', c_\mu''') \geq 0$ such that

$$\widetilde{c_m}''\|v_\Omega - w_\Omega\|^{\overline{m}}_{\overline{m},\alpha} + \widetilde{c_\mu}''\|v_\Gamma - w_\Gamma\|^{\overline{\mu}}_{\overline{\mu},\beta,\Gamma_1} \leq C|v-w|^2_{H^0} + \langle B(v) - B(w), v-w\rangle_W \tag{4.22}$$

for any $v = (v_\Omega, v_\Gamma), w = (w_\Omega, w_\Gamma) \in W$, where $\widetilde{c_m}'' > 0$ provided $m > r_\Omega$ and $\widetilde{c_\mu}'' > 0$ provided $\mu > r_\Gamma$. Then, using part (ii) of the statement, the energy identity (3.32) and (4.22) we get that $u_n' \to u'$ in $L^{\overline{m}}(0, T^*; L^{2,\overline{m}}_\alpha(\Omega))$ provided $m > r_\Omega$ and $(u_{n|\Gamma})' \to (u_{|\Gamma})'$ in $L^{\overline{\mu}}(0, T^*; L^{2,\overline{\mu}}_\beta(\Gamma_1))$ provided $\mu > r_\Gamma$. Since these conclusions are automatic when $m \leq r_\Omega$ and $\mu \leq r_\Gamma$ we get $u_n' \to u'$ in $Z(0, T^*)$.

Finally, when $(u_{0n}, u_{1n}) \to (u_0, u_1)$ in $H^{1,\rho,\theta}_{\alpha,\beta} \times H^0$ for some $\rho, \theta$ satisfying (1.13), we recall (3.27) and we note that $u_n' \to u'$ in $Z(0,T^*)$ and $u_{0n} \to u_0$ in $H^{1,\rho,\theta}_{\alpha,\beta}$ yields by a simple integration in time that $u_n \to u$ in $W^{1,1}(0, T^*; H^{1,\rho,\theta}_{\alpha,\beta}) \hookrightarrow C([0,T^*], H^{1,\rho,\theta}_{\alpha,\beta})$, concluding the proof. □

**Proof of Theorem 3.1.** It follows immediately by applying Theorem 3.2 with $F = (\widehat{f}, \widehat{g})$, which satisfies (3.29) as a consequence of (FG1–2) and (1.8). □

**Proof of Theorems 1.1 and 1.2.** They are particular cases of Theorem 3.1, by using Remarks 3.1, 3.2, 3.4 and the fact that (1.10) is trivial when $m \leq 2$ and $\mu \leq 2$. □

## 5. Regularity results

This section is devoted to make explicit, when we are dealing with problem (1.1), so $F = (\widehat{f}, \widehat{g})$, the meaning of strong solutions of problem (3.28) found in Theorem 3.2. In this way we shall get our main regularity result for problem (1.1). We shall from now on assume that

$$\Gamma \text{ is } C^2 \text{ and } \overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset. \tag{5.1}$$

Recalling the discussion made in Section 2 on Sobolev spaces on compact $C^k$ manifolds, we remark that by the arguments used in [42, pp. 38-42], when $M$ is a $C^2$ compact manifold, then

$$C^2(M) \quad \text{is dense in } W^{s,\rho}(M) \qquad \text{for } -2 \le s \le 2 \quad \text{and } \rho \in (1,\infty); \tag{5.2}$$

$$[W^{s,\rho}(M]' \simeq W^{-s,\rho'}(M) \qquad \text{for } -2 \le s \le 2 \quad \text{and } \rho \in (1,\infty). \tag{5.3}$$

Since by (5.1), $\Gamma$, $\Gamma_0$ and $\Gamma_1$ are compact $C^2$ manifolds, (5.2) and (5.3) hold true when $M = \Gamma$ and $M = \Gamma_i$, $i = 0, 1$.

We now recall some fact on the Laplace-Beltrami operator $\Delta_M$, which we shall use when $M = \Gamma$ and $M = \Gamma_i$, $i = 0, 1$, referring to [57] for more details and proofs, given there for smooth manifolds. One easily sees that the $C^2$ regularity of $M$ and the $C^1$ regularity of $(\cdot,\cdot)_M$ are enough. Then $\Delta_M$ can be at first defined on $C^2(M)$ by the formula

$$-\int_M \Delta_M u\, v = \int_M (\nabla_M u, \nabla_M v)_M \tag{5.4}$$

for any $u, v \in C^2(M)$, and $\Delta_M u = g^{-1/2}\partial_i \left(g^{ij} g^{1/2} \partial_j u\right)$ in local coordinates. Consequently $\Delta_M$ uniquely extends [12] to a bounded linear operator from $W^{s+1,\rho}(M)$ to $W^{s-1,\rho}(M)$ for any $s \in [-1, 1]$ and $1 < \rho < \infty$ (see [33, Lemma 1.4.1.3 pp. 21–24]). Since $\Delta_M 1 = 0$ the operator is not injective. The isomorphism properties of $-\Delta_M + I$ are given in Lemma B.1 in Appendix B.

Since the characteristic functions $\chi_{\Gamma_0}$ , $\chi_{\Gamma_1}$ of $\Gamma_0, \Gamma_1$ are $C^2$ on $\Gamma$, by identifying the elements of $W^{s,\rho}(\Gamma_i)$, $i = 0, 1$, with their trivial extensions to $\Gamma$ we have the decomposition

$$W^{s,\rho}(\Gamma) = W^{s,\rho}(\Gamma_0) \oplus W^{s,\rho}(\Gamma_1), \quad \text{for } \rho \in (1,\infty),\ -2 \le s \le 2, \tag{5.5}$$

so in particular $W^{s,\rho}(\Gamma_1) = \{u \in W^{s,\rho}(\Gamma) : u = 0 \text{ in } \Gamma_0\}$, coherently with (1.4). By (5.5) we also have $\Delta_\Gamma = \Delta_{\Gamma_0} + \Delta_{\Gamma_1}$, hence $\Delta_\Gamma u = \Delta_{\Gamma_1} u$ for $u \in W^{s,\rho}(\Gamma_1)$.

We recall here some classical facts on the distributional normal derivative. For any $u \in W^{1,\rho}(\Omega)$, $1 < \rho < \infty$, such that $-\Delta u = h \in L^\rho(\Omega)$ in the sense of distributions, we set $\partial_\nu u \in W^{-1/\rho,\rho}(\Gamma)$ by [13]

$$\langle \partial_\nu u, \psi\rangle_{W^{1-1/\rho',\rho'}(\Gamma)} = -\int_\Omega h\mathbb{D}\psi + \int_\Omega \nabla u \nabla(\mathbb{D}\psi) \quad \text{for all } \psi \in W^{1-1/\rho',\rho'}(\Gamma). \tag{5.6}$$

[12] here we are implicitly considering $\Delta_M$ as the restriction to real–valued distributions of the same operator acting on Sobolev spaces of complex–valued distributions, which will be studied in Appendix B.

[13] $\mathbb{D}$ was defined in subsection 2.2

The operator $u \mapsto \partial_\nu u$ is linear and bounded from $D_\rho(\Delta) = \{u \in W^{1,\rho}(\Omega) : \Delta u \in L^\rho(\Omega)\}$, equipped with the graph norm, to $W^{-1/\rho,\rho}(\Gamma)$. Moreover, since for any $\Psi \in W^{1,\rho'}(\Omega)$ such that $\Psi_{|\Gamma} = \psi$ we have $\Psi - \mathbb{D}\psi \in W_0^{1,\rho'}(\Omega)$, (5.6) extends to

$$\langle \partial_\nu u, \psi \rangle_{W^{1-1/\rho',\rho'}(\Gamma)} = -\int_\Omega h\Psi + \int_\Omega \nabla u \nabla \Psi \quad \text{for all } \psi \in W^{1-1/\rho',\rho'}(\Gamma). \tag{5.7}$$

Moreover, by (5.5), we have $\partial_\nu u = \partial_\nu u_{|\Gamma_0} + \partial_\nu u_{|\Gamma_1}$ and $\psi = \psi_{|\Gamma_0} + \psi_{|\Gamma_1}$, where $\partial_\nu u_{|\Gamma_i} \in W^{-1/\rho,\rho}(\Gamma_i)$, $\psi_{|\Gamma_i} \in W^{1-1/\rho',\rho'}(\Gamma_i)$, $i = 0, 1$, and by (5.3),

$$\langle \partial_\nu u, \psi \rangle_{W^{1-1/\rho',\rho'}(\Gamma)} = \sum_{i=0}^{1} \langle \partial_\nu u_{|\Gamma_i}, \psi_{|\Gamma_i} \rangle_{W^{1-1/\rho',\rho'}(\Gamma_i)} \tag{5.8}$$

for all $\psi \in W^{1-1/\rho',\rho'}(\Gamma)$. Hence, in particular,

$$\langle \partial_\nu u_{|\Gamma_1}, \psi \rangle_{W^{1-1/\rho',\rho'}(\Gamma_1)} = -\int_\Omega h\Psi + \int_\Omega \nabla u \nabla \Psi \tag{5.9}$$

for all $\psi \in W^{1-1/\rho',\rho'}(\Gamma_1)$ and all $\Psi \in W^{1,\rho'}(\Omega)$ such that $\Psi_{|\Gamma} = \psi$. Finally, when $u \in W^{2,\rho}(\Gamma)$ the so–defined normal derivatives coincide with the ones given by the already recalled trace theorem, that is $\partial_\nu u \in W^{2-1/\rho,\rho}(\Gamma)$ and $\partial_\nu u_{|\Gamma_i} \in W^{2-1/\rho,\rho}(\Gamma_i)$ , $i = 0, 1$.

Our main regularity result is the following one.

**Theorem 5.1.** *Suppose that (FG1–2) , (PQ1–3),* (1.8) *and* (5.1) *hold true, and let* $l, \lambda$ *be the exponents defined in* (1.15)*. Then, if*

$$(u_0, u_1) \in W^{2,l} \times X, \quad -\Delta u_0 + \widehat{P}(u_1) \in L^2(\Omega), \quad \partial_\nu u_{0|\Gamma_1} - \Delta_\Gamma u_0 + \widehat{Q}(u_{1|\Gamma}) \in L^2(\Gamma_1),$$

*the weak maximal solution* $u$ *of problem* (1.1) *found in Theorem 3.1 enjoys the further regularity*

$$u \in L^\lambda([0, T_{max}); W^{2,l}) \cap C^1_w([0, T_{max}); H^1) \cap W^{2,\infty}_{loc}([0, T_{max}); H^0), \tag{5.10}$$

$$u' \in C_w([0, T_{max}); X). \tag{5.11}$$

*Moreover*

$$u_{tt} - \Delta u + \widehat{P}(u_t) = \widehat{f}(u) \qquad \text{in } L^l(\Omega), \text{ a.e. in } (0, T_{max}), \tag{5.12}$$

$$(u_{|\Gamma})_{tt} + \partial_\nu u_{|\Gamma_1} - \Delta_\Gamma u_{|\Gamma} + \widehat{Q}((u_{|\Gamma})_t) = \widehat{g}(u_{|\Gamma}) \text{ in } L^l(\Gamma_1), \text{ a.e. on } (0, T_{max}). \tag{5.13}$$

*Remark* 5.1. By (5.11) one easily sees, integrating in time, that when $u_0 \in W^{2,l} \cap X$ then $u \in C^1_w([0, T_{\max}); X)$.

*Remark* 5.2. By (5.10)–(5.11) $u$ and all terms in (5.12) possess a representative in $L^1_{\text{loc}}((0, T_{\max}) \times \Omega)$ and all derivatives in it are actually derivatives in the sense of distributions in $(0, T_{\max}) \times \Omega$. The same remarks [14] apply to $u_{|\Gamma}$ and all terms in (5.13) in $(0, T_{\max}) \times \Gamma_1$, and one easily proves that (5.12)–(5.13) are equivalent to $u_{tt} - \Delta u + P(\cdot, u_t) = f(\cdot, u)$ a.e. in $(0, T_{\max}) \times \Omega$ and $(u_{|\Gamma})_{tt} + \partial_\nu u_{|\Gamma_1} - \Delta_\Gamma u_{|\Gamma} + Q(\cdot, (u_{|\Gamma})_t) = g(\cdot, u_{|\Gamma})$, a.e. on $(0, T_{\max}) \times \Gamma_1$.

Before proving Theorem 5.1 we characterize the domain of the operator $\mathcal{A}$ in (4.3).

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[14] by the way a distribution in $(0, T_{\max}) \times \Gamma_1$ is an elements of the dual of $C^2_c((0, T_{\max}) \times \Gamma_1)$

**Lemma 5.1.** *Let (PQ1–3) and* (5.1) *hold. Then* $D(\mathcal{A}) = D_1$*, where*

$D_1 := \{(u, v) \in W^{2,l} \times X : -\Delta u + \widehat{P}(v) \in L^2(\Omega),\ \partial_\nu u_{|\Gamma_1} - \Delta_\Gamma u + \widehat{Q}(v) \in L^2(\Gamma_1)\},$

*and*

$$Au + B(v) = (-\Delta u + \widehat{P}(v), \partial_\nu u_{|\Gamma_1} - \Delta_\Gamma u + \widehat{Q}(v)) \quad \text{for all } (u, v) \in D_1. \tag{5.14}$$

*Proof.* By (2.19), (3.8), (3.9) and (4.3) clearly $(u, v) \in D(\mathcal{A})$ if and only if $u \in H^1$, $v \in X$ and there are $h_1 \in L^2(\Omega)$, $h_2 \in L^2(\Gamma_1)$ such that, for all $\phi \in X$,

$$\int_\Omega \nabla u \nabla \phi + \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma \phi)_\Gamma + \int_\Omega \widehat{P}(u)\phi + \int_{\Gamma_1} \widehat{Q}(v)\phi = \int_\Omega h_1\phi + \int_{\Gamma_1} h_2\phi. \tag{5.15}$$

To prove that $D(\mathcal{A}) \subseteq D_1$ we fix $(u, v) \in D(\mathcal{A})$. Taking $\phi \in C_c^\infty(\Omega)$ in (5.15) we immediately get that

$$-\Delta u + \widehat{P}(v) = h_1 \qquad \text{in the sense of distributions in } \Omega. \tag{5.16}$$

We set $\widetilde{r_\Omega} = r_\Omega$ if $N \geq 3$, while $\widetilde{r_\Omega} = 2m$ if $N = 2$, so that $H^1(\Omega) \hookrightarrow L^{\widetilde{r_\Omega}}(\Omega)$. Hence, as $v \in X$, using (3.1) and Sobolev embedding we have $\widehat{P}(v) \in L^{m_1}(\Omega)$, where $m_1 := \max\{m, \widetilde{r_\Omega}\}/(m-1)$. By (5.16) then $-\Delta u \in L^{m_2}(\Omega)$ in the sense of distributions where, as $m_1 > 2$ when $N = 2$,

$$m_2 := \min\{2, m_1\} = \min\{2, \max\{m, r_\Omega\}/(m-1)\} \leq 2. \tag{5.17}$$

Since $u \in H^1(\Omega) \subset W^{1,m_2}(\Omega)$ it has a distributional derivative $\partial_\nu u_{|\Gamma_1} \in W^{-\frac{1}{m_2}, m_2}(\Gamma_1)$ and then, by (5.9), we can rewrite (5.15) as

$$\langle \partial_\nu u_{|\Gamma_1}, \phi_{|\Gamma} \rangle_{W^{1-1/m_2', m_2'}(\Gamma_1)} + \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma \phi)_\Gamma + \int_{\Gamma_1} \widehat{Q}(v)\phi = \int_{\Gamma_1} h_2\phi. \tag{5.18}$$

for all $\phi \in X$ such that $\phi_{|\Gamma} \in W^{1-1/m_2', m_2'}(\Gamma_1)$. Since for any $\psi \in C^2(\Gamma_1) \subset W^{1,2N}(\Gamma_1)$ we have $\mathbb{D}\psi \in W^{1,2N}(\Omega) \subset C(\overline{\Omega})$ , by Morrey's theorem, so $\mathbb{D}\psi \in X$, from (5.18) we can conclude that

$$\partial_\nu u_{|\Gamma_1} - \Delta_\Gamma u + \widehat{Q}(v) = h_2 \qquad \text{in } [C^2(\Gamma_1)]'. \tag{5.19}$$

We now set $\widetilde{r_\Gamma} = r_\Gamma$ if $N = 2$ and $N \geq 4$, while $\widetilde{r_\Gamma} = 2\mu$ if $N = 3$, so that $H^1(\Gamma_1) \hookrightarrow L^{\widetilde{r_\Gamma}}(\Gamma_1)$. Hence, as $v \in X$, using (3.2) and Sobolev embedding we have $\widehat{Q}(v) \in L^{\mu_1}(\Gamma_1)$, where $\mu_1 := \max\{\mu, \widetilde{r_\Gamma}\}/(\mu - 1)$. By (5.19) then $\partial_\nu u_{|\Gamma_1} - \Delta_\Gamma u = h_3$ in the sense of $[C^2(\Gamma_1)]'$ where $h_3 \in L^{\mu_2}(\Gamma_1)$ and, as $\mu_1 > 2$ when $N = 3$,

$$\mu_2 := \min\{2, \mu_1\} = \min\{2, \max\{\mu, r_\Gamma\}/(\mu - 1)\} \leq 2. \tag{5.20}$$

Since $\partial_\nu u_{|\Gamma_1} \in W^{-1/m_2, m_2}(\Gamma_1)$ and, by (1.15), (5.17) and (5.20) we have $l = \min\{m_2, \mu_2\}$, we get $-\Delta_{\Gamma_1} u \in W^{-1/l, l}(\Gamma_1)$. By Lemma B.1 then we get $u \in W^{2-1/l, l}(\Gamma_1)$. Since $-\Delta u \in L^l(\Omega)$ as $l \leq m_2$ we then get by elliptic regularity (see [33, Theorem 2.4.2.5 p. 124]) that $u \in W^{2,l}(\Omega)$, so (5.16) holds in $L^l(\Omega)$ and $\partial_\nu u_{|\Gamma_1} \in W^{1-1/l, l}(\Gamma_1)$ by the trace theorem. Plugging this information in (5.19) we then get, as $l \leq \mu_2$, that $-\Delta_{\Gamma_1} u \in L^l(\Gamma_1)$ so (5.19) holds true in this space and, by a further application of Lemma B.1 then $u \in W^{2,l}(\Gamma_1)$, proving that $D(\mathcal{A}) \subseteq D_1$.

To prove that $D_1 \subseteq D(\mathcal{A})$ let $(u, v) \in D_1$. We denote $h_1 = -\Delta u + \widehat{P}(v) \in L^2(\Omega)$ and $h_2 = \partial_\nu u_{|\Gamma_1} - \Delta_\Gamma u + \widehat{Q}(v) \in L^2(\Gamma_1)$. Since $\widehat{P}(v) \in L^l(\Omega)$ as $l \leq m_1$ we have

$$\int_\Omega -\Delta u\phi + \int_\Omega \widehat{P}(v)\phi = \int_\Omega h_1\phi \qquad \text{for all } \phi \in L^{l'}(\Omega).$$

We now point out that the classical integration by parts formula

$$\int_\Omega \nabla h \nabla k + \int_\Omega \Delta h k = \int_\Gamma \partial_\nu h\, k \tag{5.21}$$

which is standard when $h \in H^2(\Omega)$ and $k \in H^1(\Omega)$ (see [33, Lemma 1.5.3.7 p.59]) extends to $h \in W^{2,l}(\Omega) \cap H^1(\Omega)$ and $k \in H^{1,l',l'}_{1,1}$. Indeed, by using [1, Theorem 4.26, p. 84], $h$ can be approximated in $W^{2,l}(\Omega) \cap H^1(\Omega)$ by a sequence $(h_n)$ in $C^2(\overline{\Omega}) \subset H^2(\Omega)$, so we can pass to the limit in (5.21) as $k \in L^{l'}(\Omega)$ and $k_{|\Gamma} \in L^{l'}(\Gamma)$. Hence we get that (5.15) holds for all $\phi \in H^{1,l',l'}_{1,1}$. By Lemma 2.1 then (5.15) holds for all $\phi \in X$, so proving that $D_1 \subseteq D(\mathcal{A})$ and (5.14) holds, concluding the proof. □

*Proof of Theorem 5.1.* By Lemma 5.1 we have $(u_0, u_1) \in D(\mathcal{A})$, hence by Theorem 3.2 the maximal solution of (1.1) is actually a strong solution of (3.28) when $F = (\widehat{f}, \widehat{g})$, so $u \in W^{1,\infty}_{\rm loc}([0, T_{\max}); H^1) \cap W^{2,\infty}_{\rm loc}([0, T_{\max}); H^0)$, $(u(t), u'(t)) \in D_1$ for all $t \in [0, T_{\max})$ and, by (5.14), (5.12)–(5.13) hold true.

To prove (5.10) we note that, since $\widehat{P}(u_t) \in L^\infty_{\rm loc}([0, T_{\max}); L^{\widetilde{r_\Omega}/(m-1)}(\Omega))$ when $m \le \widetilde{r_\Omega}$ and $\widehat{P}(u_t) \in L^{m'}_{\rm loc}([0, T_{\max}); L^{m'}(\Omega))$ when $m > \widetilde{r_\Omega}$, we have

$$\widehat{P}(u_t) \in L^{\lambda_1}_{\rm loc}([0, T_{\max}); L^{m_1}(\Omega)), \tag{5.22}$$

where $\lambda_1 = \infty$ when $m \le r_\Omega$ and $\lambda_1 = m'$ otherwise. Since moreover $u_{tt}, \widehat{f} \in L^\infty_{\rm loc}([0, T_{\max}); L^2(\Omega))$ we then get from (5.12) that

$$\Delta u \in L^{\lambda_1}_{\rm loc}([0, T_{\max}); L^{m_2}(\Omega)) \subset L^{\lambda}_{\rm loc}([0, T_{\max}); L^l(\Omega)). \tag{5.23}$$

Consequently, by the boundedness of the distributional normal derivatives, $\partial_\nu u|_{|\Gamma_1} \in L^{\lambda}_{\rm loc}([0, T_{\max}); W^{-1/l,l}(\Gamma_1))$. Since $\widehat{Q}((u_{|\Gamma})_t \in L^\infty_{\rm loc}([0, T_{\max}); L^{\widetilde{r_\Gamma}/(\mu-1)}(\Gamma_1))$ when $\mu \le \widetilde{r_\Gamma}$ and $\widehat{Q}((u_{|\Gamma})_t \in L^{\mu'}_{\rm loc}([0, T_{\max}); L^{\mu'}(\Gamma_1))$ when $\mu > \widetilde{r_\Gamma}$, we have

$$\widehat{Q}((u_{|\Gamma})_t \in L^{\lambda_2}_{\rm loc}([0, T_{\max}); L^{\mu_1}(\Gamma_1)), \tag{5.24}$$

where $\lambda_2 = \infty$ when $\mu \le r_\Gamma$ and $\lambda_2 = \mu'$ otherwise.

Since moreover $(u_{|\Gamma})_{tt}, \widehat{g} \in L^\infty_{\rm loc}([0, T_{\max}); L^2(\Gamma))$ we then get from (5.13) that $\Delta_{\Gamma_1} u_{|\Gamma_1} \in L^{\lambda}_{\rm loc}([0, T_{\max}); W^{-1/l,l}(\Gamma_1))$. As $l \le 2$, from Lemma B.1 we get $u_{|\Gamma_1} \in L^{\lambda}_{\rm loc}([0, T_{\max}); W^{2-1/l,l}(\Gamma_1))$. By (5.23) and the already quoted elliptic regularity result we have $u \in L^{\lambda}_{\rm loc}([0, T_{\max}); W^{2,l}(\Omega))$, and consequently we get $\partial_\nu u|_{|\Gamma_1} \in L^{\lambda}_{\rm loc}([0, T_{\max}); W^{1-1/l,l}(\Gamma_1))$. Using (5.13) again then we get

$$\Delta_{\Gamma_1} u \in L^{\lambda}_{\rm loc}([0, T_{\max}); L^l(\Gamma_1)),$$

hence by Lemma B.1 we have $u_{|\Gamma_1} \in L^{\lambda}_{\rm loc}([0, T_{\max}); W^{2,l}(\Gamma_1))$.

Since $u_t \in C([0, T_{\max}); H^0) \cap L^\infty_{\rm loc}(0, T_{\max}; H^1)$, by [55, Theorem 2.1 p.544] and Lemma 2.2 we then get $u_t \in C_w([0, T_{\max}); H^1)$, completing the proof of (5.10).

To prove (5.11) we remark that, as shown is the proof of Lemma 4.2 – Step 1, we have $\langle B(u'), u' \rangle_w \in L^\infty_{\rm loc}([0, T_{\max}))$, hence by (PQ3) we have $u' \in L^\infty_{\rm loc}([0, T_{\max}); W)$ and consequently using Lemma 2.2 and the result by W. Strauss already recalled we get (5.11), completing the proof. □

When the damping terms are not supersupercritical, the time regularity (5.10) can be improved as follows.

**Corollary 5.1.** *Suppose that all assumptions in Theorem 5.1 hold true, and moreover suppose*

$$1 < m \le r_{\Omega} \qquad \textit{and} \quad 1 < \mu \le r_{\Gamma}. \tag{5.25}$$

*Then, if*

$$(u_0,u_1) \in W^{2,l} \times H^1, \quad -\Delta u_0 + \widehat{P}(u_1) \in L^2(\Omega), \quad \partial_\nu u_{0|\Gamma_1} - \Delta_\Gamma u_0 + \widehat{Q}(u_{1|\Gamma}) \in L^2(\Gamma_1),$$

*the regularity* (5.10) *is improved to*

$$u \in C_w([0,T_{max}); W^{2,l}) \cap C^1_w([0,T_{max}); H^1) \cap C^2_w([0,T_{max}); H^0). \tag{5.26}$$

*Consequently, when*

$$1 < m \le 1 + \frac{r_\Omega}{2} \qquad \textit{and} \quad 1 < \mu \le \frac{r_\Gamma}{2}, \tag{5.27}$$

*for initial data* $(u_0,u_1) \in H^2 \times H^1$ *we have*

$$u \in C_w([0,T_{max}); H^2) \cap C^1_w([0,T_{max}); H^1) \cap C^2_w([0,T_{max}); H^0). \tag{5.28}$$

*Proof.* When (5.25) holds, by (1.15) we have $\lambda = \infty$, hence by (5.10) we get $u \in L^\infty_{\text{loc}}([0,T_{\max}); W^{2,l})$. Since $W^{2,l}(\Omega)$ and $W^{2,l}(\Gamma_1)$ are (respectively) dense in $H^1(\Omega)$ and $H^1(\Gamma_1)$, by [55, Theorem 2.1 p.544] we get $u \in C_w([0,T_{\max}); W^{2,l})$. Hence $\Delta u \in C_w([0,T_{\max}); L^l(\Omega))$ and $\Delta_{\Gamma_1} u - \partial_\nu u_{|\Gamma_1} \in C_w([0,T_{\max}); L^l(\Gamma_1))$.

Moreover, by (5.22) and (5.24), we also have $P(u_t) \in L^\infty_{\text{loc}}([0,T_{\max}); L^l(\Omega))$ and $Q((u_\Gamma)_t) \in L^\infty_{\text{loc}}([0,T_{\max}); L^l(\Gamma_1))$. Hence, as $\widehat{f}(u) \in C_w([0,T_{\max}]; L^2(\Omega))$ and $\widehat{g}(u_{|\Gamma}) \in C_w([0,T_{\max}]; L^2(\Gamma_1))$, by (5.12)–(5.13) we get $u'' \in C_w([0,T_{\max}); L^l(\Omega) \times L^l(\Gamma_1))$. Hence by (5.10), the density of $H^0$ in $L^l(\Omega) \times L^l(\Gamma_1)$ and [55, Theorem 2.1 p.544] again we get $u'' \in C_w([0,T_{\max}); H^0)$, concluding the proof of (5.26).

When (5.27) holds we also have $l = 2$ and for data $(u_0,u_1) \in H^2 \times H^1$ the conditions $-\Delta u_0 + \widehat{P}(u_1) \in L^2(\Omega)$ and $\partial_\nu u_{0|\Gamma_1} - \Delta_\Gamma u_0 + \widehat{Q}(u_{1|\Gamma}) \in L^2(\Gamma_1)$ are automatic, so (5.28) holds. ☐

**Proof of Theorems 1.3 and 1.4.** By Remarks 3.1, 3.2 and 3.4 they are particular cases of Theorem 5.1 and Corollary 5.1. ☐

## 6. Global existence versus blow–up

This section is devoted to our global existence and blow–up results for problem (1.1). Before giving them we need some preliminaries. We shall assume in the sequel that assumptions (PQ1–3), (FG1–2) and (1.8) hold true.

6.1. **The energy function.** To use energy methods it is fruitful to introduce an energy function involving the potential operator associated to $F = (\widehat{f}, \widehat{g})$, and to write the energy identity (3.26) in terms of this function.

We first need to point out the following abstract version of the classical chain rule, which easy proof is given for the reader's convenience.

**Lemma 6.1.** *Let $V_0$ and $H_0$ be real Hilbert spaces, $I$ be a real interval and denote by $(\cdot,\cdot)_{H_0}$ the scalar product of $H_0$. Suppose that $V_0 \hookrightarrow H_0$ with dense embedding, so that $V_0 \hookrightarrow H_0 \simeq H_0' \hookrightarrow V_0'$.*

*Then for any $J_1 \in C^1(V_0)$ such that its Frèchet derivative $J_1'$ is locally Lipschitz from $V_0$ to $H_0$ and any $w \in C(I;V_0) \cap C^1(I;H_0)$ we have $J_1 \cdot w \in C^1(I)$ and $(J_1 \cdot w)' = (J_1' \cdot w, w')_{H_0}$ in $I$, where $\cdot$ denotes the composition product.*

*Proof.* At first we note that $w$ can be trivially extended, with the same regularity, to the whole of $\mathbb{R}$, so we assume without restriction that $I = \mathbb{R}$. Next we remark that our claim reduces to the well–known chain rule for the Frèchet derivative (see [3, Proposition 1.4, p. 12] when $w \in C^1(I;V_0)$.

In the general case, denoting by $(\rho)_n$ a standard sequence of mollifiers and by $*$ the standard convolution product in $\mathbb{R}$, we set $w_n = \rho_n * w$, so $w_n \in C^1(I;V_0)$ and, by previous remark, $(J_1 \cdot w_n)' = (J_1' \cdot w_n, w_n')_{H_0}$ in $\mathbb{R}$ for all $n \in \mathbb{N}$ . Since the proof of [16, Proposition 4.21, p. 108] trivially extends to vector–valued functions, we also have $w_n \to w$ in $V_0$ and $w_n' \to w'$ in $H_0$ locally uniformly in $\mathbb{R}$. Since $J_1'$ is locally Lipschitz from $V_0$ to $H_0$ it then follows that $J_1' \cdot w_n \to J_1' \cdot w$ in $H_0$ locally uniformly in $\mathbb{R}$ and consequently $(J_1 \cdot w_n)' = (J_1' \cdot w_n, w_n')_{H_0} \to (J_1' \cdot w, w')_{H_0}$ locally uniformly in $\mathbb{R}$. Since $J_1 \in C(V_0)$ we also have $J_1 \cdot w_n \to J_1 \cdot w$ in $\mathbb{R}$. Our assertion then follows by standard results on uniformly convergent real sequences. $\square$

We introduce the primitives of the functions $f$ and $g$ by

$$\mathfrak{F}(x,u) = \int_0^u f(x,s)\,ds, \qquad \text{and} \quad \mathfrak{G}(y,u) = \int_0^u g(y,s)\,ds, \tag{6.1}$$

for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $u \in \mathbb{R}$, and we note that by (FG1) there are constants $c_p''', c_q''' \geq 0$ such that

$$|\mathfrak{F}(x,u)| \leq c_p'''(1+|u|^p), \qquad \text{and} \quad |\mathfrak{G}(y,u)| \leq c_q'''(1+|u|^q) \tag{6.2}$$

for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $u \in \mathbb{R}$.

By (1.8), (6.2) and Sobolev embedding theorem we can set the potential operator $J : H^1 \to \mathbb{R}$ by

$$J(v) = \int_\Omega \mathfrak{F}(\cdot,v) + \int_{\Gamma_1} \mathfrak{G}(\cdot, v_{|\Gamma}) \qquad \text{for all } v \in H^1. \tag{6.3}$$

By (FG1), (1.8) and Sobolev embedding theorem, using the same arguments applied to prove [3, Theorem 2.2, p. 16] one easily gets that $J \in C^1(H^1)$, its Frèchet derivative $J'$ being given by $F = (\widehat{f}, \widehat{g})$, which is locally Lipschitz from $H^1$ to $H^0$.

Hence, by (2.24) and Lemma 6.1, for any weak solution $u$ of (1.1) we have $J \cdot u \in C^1(\text{dom } u)$ and

$$(J \cdot u)' = \int_\Omega f(\cdot,u)u_t + \int_{\Gamma_1} g(\cdot, u_{|\Gamma})(u_{|\Gamma})_t. \tag{6.4}$$

We also introduce the energy functional $\mathcal{E} \in C^1(\mathcal{H})$ defined by

$$\mathcal{E}(v,w) = \frac{1}{2}\|w\|_{H^0}^2 + \tfrac{1}{2}\int_\Omega |\nabla v|^2 + \tfrac{1}{2}\int_{\Gamma_1} |\nabla_\Gamma v|_\Gamma^2 - J(v), \quad \text{for all } (v,w) \in \mathcal{H}, \tag{6.5}$$

and the energy function associated to a weak maximal solution $u$ of (1.1) by

$$\mathcal{E}_u(t) = \mathcal{E}(u(t), u'(t)), \qquad \text{for all } t \in [0, T_{\max}). \tag{6.6}$$

By (6.4) and (6.5) the energy identity (3.26) can be rewritten as

$$\mathcal{E}_u(t) - \mathcal{E}_u(s) + \int_s^t \langle B(u'), u' \rangle_W = 0 \quad \text{for all } s, t \in [0, T_{\max}). \tag{6.7}$$

Consequently, by Lemma 3.1, (2.9), (6.5) and (6.6), $\mathcal{E}_u$ is decreasing and we have

$$\begin{aligned} \frac{1}{2}\|u'(t)\|_{H^0}^2 + \frac{1}{2}\|u(t)\|_{H^1}^2 =& \mathcal{E}_u(0) - \int_0^t \langle B(u'), u' \rangle_W + \frac{1}{2}\|u(t)\|_{H^0}^2 + J(u(t)) \\ \le& \mathcal{E}_u(0) + \frac{1}{2}\|u(t)\|_{H^0}^2 + J(u(t)) \quad \text{for } t \in [0, T_{\max}). \end{aligned} \tag{6.8}$$

The alternative between global existence and blow–up depends on the specific structure of the nonlinearities involved. We shall separately treat two different cases.

6.2. **Blow–up when damping terms are linear.** We shall consider in this subsection damping terms satisfying the following assumption

(PQ5) there are $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Gamma_1)$, $\alpha, \beta \ge 0$, such that

$$P(x, v) = \alpha(x)v, \qquad \text{and} \quad Q(y, v) = \beta(y)v$$

for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $v \in \mathbb{R}$.

*Remark* 6.1. Trivially (PQ5) implies assumptions (PQ1–3) with $m = \mu = 2$, and in some sense override them. Moreover in the case considered in problem (1.2) it reduces to assumption (I)$'$.

Moreover we shall consider in this subsection source terms satisfying the following specific assumption:

(FG4) at least one between $f$ and $g$ is not a.e. vanishing and there are exponents $\overline{p}, \overline{q} > 2$ such that, for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $u \in \mathbb{R}$,

$$f(x, u)u \ge \overline{p}\,\mathfrak{F}(x, u) \ge 0, \qquad \text{and} \quad g(y, u)u \ge \overline{q}\,\mathfrak{G}(y, u) \ge 0. \tag{6.9}$$

*Remark* 6.2. In the case considered in problem (1.2), i.e. $f(x, u) = f_0(u)$ and $g(x, u) = g_0(u)$, assumption (FG4) reduces to assumption (IV). Another specific example is is given by (3.23) when $f_0$ and $g_0$ still satisfy (IV).

We can now give our blow–up result

**Theorem 6.1.** *Let (PQ4), (FG1–2), (FG4),* (1.8) *hold. Then*

(i) *$N = \{(u_0, u_1) \in \mathcal{H} : \ \mathcal{E}(u_0, u_1) < 0\} \ne \emptyset$, and*
(ii) *for any $(u_0, u_1) \in N$ the unique maximal weak solution $u$ of* (1.1) *blows–up in finite time, that is $T_{\max} < \infty$, and* (1.28) *holds.*

*Proof.* We first prove (ii). By (PQ5), we have $X = H^1$ and $W = H^0$. Since we are going to apply [38, Theorem 1] to (1.1), in the sequel we are going to check its assumptions. Referring to the notation of the quoted paper, adding a $_*$ subscript to it, we set

$$V_* = H^0, \qquad Y_* = H^0, \qquad W_* = H^1, \qquad X_* = L^p(\Omega) \times L^q(\Gamma_1)$$

where, according to (1.4), $L^q(\Gamma_1)$ is identified with $\{v \in L^q(\Gamma) : v = 0 \text{ on } \Gamma_0\}$. We note that, by (1.8) and Sobolev embedding theorem we have $W_* \hookrightarrow X_*$.

We also set the operators

$$P_* : V_* \to V_*',\ A_* : W_* \to W_*',\ F_* : X_* \to X_*',\ Q_* : [0,\infty) \times Y_* \to Y_*'$$

by

$$P_* = \mathrm{Id}, \qquad A_* = A, \qquad F_* = F, \qquad Q_* = B,$$

nothing that in our case $A_*$, $F_*$ and $Q_*$ are autonomous so no explicit dependence on time is needed. Trivially $P_*$ and $Q_*$ are non–negative definite and symmetric. Moreover $A_*$ and $F_*$ are the Frèchet derivatives of the $C^1$ potentials $\mathcal{A}_* : W_* \to \mathbb{R}$, $\mathcal{F}_* : X_* \to \mathbb{R}$ respectively given for all $v \in W_*$, $(v_1, v_2) \in X_*$ by

$$\mathcal{A}_*(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|\nabla_\Gamma v\|_\Gamma^2 \quad \text{and} \quad \mathcal{F}_*(v_1, v_2) = \int_\Omega \mathfrak{F}(\cdot, v_1) + \int_{\Gamma_1} \mathfrak{G}(\cdot, v_2).$$

With this setting the abstract evolution equation

$$P_* u'' + Q_* u' + A_*(u) = F_*(u) \tag{6.10}$$

considered in [38, (2.1)] formally reduces to (3.28) and, taking $G_* = W_*$ as the nontrivial subspace of $V_*$, $W_*$ and $Y_*$ we have $K_* = C([0,\infty); H^1) \cap C^1([0,\infty); H^0)$, hence, by Definition 3.1 and (3.26) strong solutions of (6.10) in the sense of [38] exactly reduce to weak solutions of (1.1) in $[0,\infty)$. Moreover, to check the specific assumptions of [38, Theorem 1] we note that, by (FG4) for all $v \in G_* = H^1$ we have

$$\begin{aligned}
\langle A_*(v), v\rangle_{W_*} - \langle F_*(v), v\rangle_{X_*} =& \|\nabla v\|_2^2 + \|\nabla_\Gamma v\|_\Gamma^2 - \int_\Omega f(\cdot, v) - \int_{\Gamma_1} g(\cdot, v) \\
\le& 2\mathcal{A}_*(v) - \overline{p}\int_\Omega \mathfrak{F}(\cdot, v) - \overline{q}\int_{\Gamma_1} \mathfrak{G}(\cdot, v) \\
\le& q_* \left[\mathcal{A}_*(v) - \mathcal{F}_*(v)\right]
\end{aligned}$$

where $q_* := \min\{\overline{p}, \overline{q}\} > 2$, hence [38, (2.3)] holds, while [38, (2.4)] is trivially satisfied since $\mathcal{A}_*$ and $\mathcal{F}_*$ does not depend on $t$.

By the quoted result we then get that (1.1) has no global weak solutions in $[0,\infty)$ with $(u_0, u_1) \in N$, and then $T_{\max} < \infty$. By Theorem 3.1 we get the first limit in (1.28). Consequently, by (6.2) and (6.8), since $p, q \ge 2$, we also get the second limit in (1.28).

We now prove (i), first considering the case in which $g$ does not vanish a.e. in $\Gamma_1 \times \mathbb{R}$ (so $\Gamma_1 \neq \emptyset$). Since $g(x,u)u \ge 0$ at least one between the two sets

$$E^\pm = \{(x,u) \in \Gamma_1 \times \mathbb{R} : \pm g(x,u) > 0,\ \pm u > 0\} \tag{6.11}$$

has positive measure in $\Gamma_1 \times \mathbb{R}$. In the sequel of the proof the symbol $\pm$ means $+$ is $E^+$ has positive measure and $-$ if $E^-$ has positive measure. Hence there are $C \subseteq \Gamma_1$, $\varepsilon > 0$ and $\overline{u} \in \mathbb{R}$ such that $\pm\overline{u} > 0$, $C \times (\overline{u} - \varepsilon, \overline{u} + \varepsilon) \subseteq E^\pm$ and $\sigma(C) > 0$. We denote

$$B_1 = \{x' \in \mathbb{R}^{N-1} : |x'| < 1\}, \quad Q = B_1 \times (-1,1), \quad Q^+ = B_1 \times (0,1), \quad Q^0 = B_1 \times \{0\}.$$

Since $\Gamma$ is $C^1$ and compact there is an open set $U_0$ in $\mathbb{R}^N$ and a coordinate map $\psi_1 : Q \to U_0$, bijective and such that $\psi_1 \in C^1(\overline{Q})$, $\psi_1^{-1} \in C^1(\overline{U_0})$, $\psi_1(Q^+) = U_0 \cap \Omega$, $\psi_1(Q^0) = U_0 \cap \Gamma$ and $\sigma(U_0 \cap C) > 0$. We denote $\psi_2 = \psi_1(\cdot, 0) : B_1 \to \Gamma$, $\psi_2 \in C^1(\overline{B_1})$ and $D = \psi_2^{-1}(U_0 \cap C)$. Since $\sigma(U_0 \cap C) = \int_D |\partial_{x_1}\psi_2 \wedge \ldots \wedge \partial_{x_{N-1}}\psi_2|\, dx'$,

where $x' = (x_1, \dots, x_{N-1})$, $D$ has positive measure in $R^{N-1}$ and hence it contains an open ball $B_2$ of radius $r > 0$. We set $B = \psi_2(B_2) \subseteq U_0 \cap C \subseteq \Gamma_1$ and $U_1 = \psi_1(B_2 \times (-1, 1))$. Hence $U_1$ is open in $\mathbb{R}^N$ and $U_1 \cap \Gamma_0 = B \cap \Gamma_1 = \emptyset$.

By (6.1) and (6.11), since $B \times (\overline{u} - \varepsilon, \overline{u} + \varepsilon) \subseteq E^\pm$, we get that $\phi_2 := \mathfrak{G}(\cdot, \overline{u}) > 0$ a.e. in $B$. Integrating the differential inequality in (6.9) from 0 to $\overline{u}$ and denoting $\phi_3 = \phi_2 |\overline{u}|^{-\overline{q}}$, we get $\mathfrak{G}(\cdot, u) \geq \phi_3 |u|^{\overline{q}}$ a.e. in $B$ when $u - \overline{u} \in \mathbb{R}^\pm_0$, and consequently

$$\mathfrak{G}(\cdot, u) \geq \phi_3 |u|^{\overline{q}} - \phi_2, \qquad \text{a.e. in } B, \text{ when } u \in \mathbb{R}^\pm_0. \tag{6.12}$$

Now (see [16, p. 210]) we fix $\eta_0 \in C^\infty(\mathbb{R})$ such that $\eta_0(s) = 1$ if $s < 1/4$ and $\eta_0(s) = 0$ if $s > 3/4$. Moreover we denote $\overline{w_0}(x', x_N) = \eta_0(|x'|/r)\, \eta_0(|x_N|)$ for $(x', x_N) \in B_2 \times (-1, 1)$ and $w_0 = \overline{w_0} \cdot \psi^{-1}$. Hence $w_0 \geq 0$, $w_{0|\Gamma_1} \not\equiv 0$, $w_0 \in C^1_c(U_1)$ and then, as $U_1 \cap \Gamma_0 = \emptyset$, $w_0 \in H^1$. Hence, by (6.5), (FG4) and (6.12) we have

$$\mathcal{E}(sw_0, u_1) \leq \tfrac{1}{2}\|u_1\|^2_{H^0} + \tfrac{1}{2}\left(\|\nabla w_0\|_2^2 + \|\nabla_\Gamma w_0\|^2_{2,\Gamma_1}\right) s^2 - \left(\int_B \phi_3\, |w_0|^{\overline{q}}\right) |s|^{\overline{q}} + \|\phi_2\|_{1,\Gamma_1}$$

for all $u_1 \in H^0$ and $s \in \mathbb{R}^\pm_0$. Since $\overline{q} > 2$ and $\int_B \phi_3\, |w_0|^{\overline{q}} > 0$ it follows that $\mathcal{E}(sw_0, u_1) \to -\infty$ when $s \to \pm\infty$ and $u_1$ is fixed. Hence, choosing $u_0 = sw_0$ for $s \in \mathbb{R}^\pm$ large enough, depending on $\|u_1\|_{H^0}$, we get $(u_0, u_1) \in N$.

When $f$ does not vanish a.e. the proof repeats the arguments used in the previous case and hence it is omitted. We just mention that we can directly take $B \subseteq \Omega$ to be an open ball of radius $r > 0$ and define $w_0 \in C^\infty_c(B)$ by $w_0(x) = \eta_0(|x|/r)$. □

**Proof of Theorem 1.5.** By Remarks 3.1, 3.4, 6.1 and 6.2, the statement is a particular case of Theorem 6.1. □

6.3. **Global existence.** In this subsection we shall deal with perturbation terms $f$ and $g$ which source part has at most linear growth at infinity, uniformly in the space variable, or, roughly, it is dominated by the corresponding damping term. More precisely we shall make the following specific assumption:

(FGQP) there are $p_1$ and $q_1$ verifying (1.29) and constants $C_{p_1}, C_{q_1} \geq 0$ such that

$$\mathfrak{F}(x, u) \leq C_{p_1}\left[1 + u^2 + \gamma_0(x)|u|^{p_1}\right], \quad \mathfrak{G}(y, u) \leq C_{q_1}\left[1 + u^2 + \delta_0(y)|u|^{q_1}\right]$$

for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $u \in \mathbb{R}$.

Since $\mathfrak{F}(\cdot, u) = \int_0^1 f(\cdot, su)u\, ds$ (and similarly $\mathfrak{G}$), assumption (FGQP) is a weak version of of the following one:

(FGQP)$'$ there are $p_1$ and $q_1$ verifying (1.29) and constants $C'_{p_1}, C'_{q_1} \geq 0$ such that

$$f(x, u)u \leq C'_{p_1}\left[|u| + u^2 + \gamma_0(x)|u|^{p_1}\right], \quad g(y, u)u \leq C'_{q_1}\left[|u| + u^2 + \delta_0(y)|u|^{q_1}\right]$$

for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $u \in \mathbb{R}$.

*Remark* 6.3. Assumptions (FG1–2) and (FGQP)$'$ hold provided

$$f = f^0 + f^1 + f^2, \qquad g = g^0 + g^1 + g^2, \tag{6.13}$$

where $f^i$, $g^i$ satisfy the following assumptions:

(i) $f^0$ and $g^0$ are a.e. bounded and independent on $u$;
(ii) $f^1$ and $g^1$ satisfy (FG1–2) with exponents $p_1$ and $q_1$ satisfying (1.29), and

(a) when $p_1 > 2$ and $\operatorname{ess\,inf}_\Omega \alpha = 0$ there is a constant $\overline{c_{p_1}} \geq 0$ such [15] that
$$|f^1(x,u)| \leq \overline{c_{p_1}} \left[1 + |u| + \alpha(x)|u|^{p_1-1}\right]$$
for a.a. $x \in \Omega$ and all $u \in \mathbb{R}$;

(b) when $q_1 > 2$ and $\operatorname{ess\,inf}_{\Gamma_1} \beta = 0$ there is a constant $\overline{c_{q_1}} \geq 0$ such that
$$|g^1(y,u)| \leq \overline{c_{q_1}} \left[1 + |u| + \beta(y)|u|^{q_1-1}\right]$$
for a.a. $y \in \Gamma_1$ and all $u \in \mathbb{R}$;

(iii) $f^2$ and $g^2$ satisfy (FG1–2), $f^2(x,u)u \leq 0$ and $g^2(y,u)u \leq 0$ for a.a. $x \in \Omega$, $y \in \Gamma_1$ and all $u \in \mathbb{R}$.

Conversely any couple of functions $f$ and $g$ satisfying (FG1–2) and (FGQP)$'$ admits a decomposition of the form (6.13)–(i–iii) with $f^1$ and $g^1$ being source terms. Indeed one can set $f^0 = f(\cdot,0)$,
$$f^1(\cdot,u) = \begin{cases} [f(\cdot,u) - f^0]^+ & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -[f(\cdot,u) - f^0]^- & \text{if } u < 0, \end{cases} \quad \text{and } f^2(\cdot,u) \begin{cases} -[f(\cdot,u) - f^0]^- & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ [f(\cdot,u) - f^0]^+ & \text{if } u < 0, \end{cases}$$
and define $g^0$, $g^1$, $g^2$ in the analogous way.

*Remark* 6.4. When dealing with problem (1.2) assumption (FGQP) reduces to (V). The function $f \equiv f_2$ defined in (3.22) satisfies (FGQP) provided one among the following cases occurs:

(i) $\gamma_1^+ = \gamma_2^+ \equiv 0$,
(ii) $\gamma_2^+ \equiv 0$, $\gamma_1^+ \not\equiv 0$, $\widetilde{p} \leq \max\{2,m\}$ and $\gamma_1 \leq c_1'\alpha$ a.e. in $\Omega$ when $\widetilde{p} > 2$
(iii) $\gamma_1^+ \not\equiv 0$, $\gamma_2^+ \not\equiv 0$, $p \leq \max\{2,m\}$, $\gamma_1 \leq c_1'\alpha$ when $\widetilde{p} > 2$ and $\gamma_2 \leq c_2'\alpha$ when $p > 2$, a.e. in $\Omega$,

where $c_1', c_2' \geq 0$ denote suitable constants. The analogous cases (j–jjj) occurs when $g \equiv g_2$, so that $(f_2, g_2)$ satisfies (FGQP) provided any combination between the cases (i–iii) and (j–jjj) occurs. In particular then a damping term can be localized provided the corresponding source is equally localized.

Finally when $f \equiv f_3$ and $g \equiv g_3$ as in (3.23), assumption (FGQP) holds provided $f_0$ and $g_0$ satisfy assumption (V) (where we conventionally take $f_0 \equiv 0$ when $\gamma \equiv 0$ and $g_0 \equiv 0$ when $\delta \equiv 0$), $\gamma \leq \alpha$ when $p_1 > 2$ and $\delta \leq \beta$ when $q_1 > 2$.

Our global existence result is the following one.

**Theorem 6.2.** *Let (PQ1–3), (FG1–2), (FGQP) and* (1.8) *hold. Then, for any couple of data $(u_0, u_1) \in \mathcal{H}$ the unique maximal weak solution $u$ of* (1.1) *is global in time, that is $T_{\max} = \infty$. Consequently the semi–flow generated by problem* (1.1) *is a dynamical system in $H^1 \times H^0$ and, when also (III) holds, in $H^{1,\rho,\theta}_{\alpha,\beta} \times H^0$ for $(\rho, \theta)$ verifying* (1.13).

*Proof.* We suppose by contradiction that $T_{\max} < \infty$, so by Theorem 3.1 we have
$$\lim_{t \to T_{\max}^-} \|u(t)\|_{H^1} + \|u'(t)\|_{H^0} = \infty. \tag{6.14}$$

[15]that is $\overline{\lim}_{|u|\to\infty} |f_1(\cdot,u)|/|u|^{p_1-1} \leq \overline{c_{p_1}}\alpha$ a.e. uniformly in $\Omega$

We introduce the functional $\mathcal{I} : H^1 \to \mathbb{R}_0^+$ given by

$$\mathcal{I}(v) = C_{p_1} \int_\Omega \alpha |v|^{p_1} + C_{q_1} \int_{\Gamma_1} \beta |v|^{q_1}. \tag{6.15}$$

Since the functions $p_1 C_{p_1}(x)|u|^{p_1-2}u$ and $q_1 C_{q_1}\beta(x)|u|^{q_1-2}u$ satisfy assumption (FG1), we see as in subsection 6.1 that $\mathcal{I} \in C^1(H^1)$, with Frèchet derivative being given by the couple of Nemitskii operators associated with them. Hence by Lemma 6.1 we have $\mathcal{I} \cdot u \in C^1([0, T_{\max}))$ and, for all $t \in [0, T_{\max})$,

$$\mathcal{I}(u(t)) - \mathcal{I}(u_0) = \int_0^t \left[ \int_\Omega p_1 C_{p_1} \alpha |u|^{p_1-2} u u_t + \int_{\Gamma_1} q_1 C_{q_1} |u|^{q_1-2} u (u_{|\Gamma})_t \right]. \tag{6.16}$$

We introduce an auxiliary function associated to $u$ by

$$\Upsilon(t) = \tfrac12 \|u'(t)\|_{H^0}^2 + \tfrac12 \|u(t)\|_{H^1}^2 + \mathcal{I}(u(t)), \qquad \text{for all } t \in [0, T_{\max}). \tag{6.17}$$

By (6.8) and (6.17) we have

$$\Upsilon(t) = \mathcal{E}_u(0) + \tfrac12 \|u(t)\|_{H^0}^2 + J(u(t)) + \mathcal{I}(u(t)) - \int_0^t \langle B(u'), u' \rangle_W. \tag{6.18}$$

By (6.15) and assumption (FGQP) we get

$$J(v) \le [C_{p_1}|\Omega| + C_{q_1}\sigma(\Gamma)] \left(1 + \|v\|_{H^0}^2\right) + \mathcal{I}(v) \qquad \text{for all } v \in H^1. \tag{6.19}$$

By (6.18)– (6.19) we thus obtain

$$\Upsilon(t) \le \mathcal{E}_u(0) + k_1 + k_1 \|u(t)\|_{H^0}^2 + 2\mathcal{I}(u(t)) - \int_0^t \langle B(u'), u' \rangle_W, \tag{6.20}$$

where $k_1 = C_{p_1}|\Omega| + C_{q_1}\sigma(\Gamma) + 1/2$. Consequently, by (6.16),

$$\begin{aligned} \Upsilon(t) \le & k_2 + \int_0^t \Big[ 2k_1 (u', u)_{H^0} - \langle B(u'), u' \rangle_W \\ & + 2p_1 C_{p_1} \int_\Omega \alpha |u|^{p_1-2} u u_t + 2q_1 C_{q_1} \int_{\Gamma_1} \beta |u|^{q_1-2} u (u_{|\Gamma})_t \Big], \end{aligned} \tag{6.21}$$

where $k_2 = \mathcal{E}_u(0) + 2\mathcal{I}(u_0) + k_1(1 + \|u_0\|_{H^0}^2)$. Consequently, by assumption (PQ3), Cauchy–Schwartz and Young inequalities, we get the preliminary estimate

$$\begin{aligned} \Upsilon(t) \le & k_2 + \int_0^t \Big[ -c_m' \|[u_t]_\alpha\|_{m,\alpha}^m - c_\mu' \|[(u_\Gamma)_t]_\beta\|_{\mu,\beta}^\mu + k_1 \|u'\|_{H^0}^2 + k_1 \|u\|_{H^0}^2 \\ & + 2p_1 C_{p_1} \int_\Omega \alpha |u_t| |u|^{p-1} |u_t| + 2q_1 C_{q_1} \int_{\Gamma_1} \beta |u|^{q-1} |(u_{|\Gamma})_t| |(u_{|\Gamma})_t| \Big] \end{aligned} \tag{6.22}$$

for all $t \in [0, T_{\max})$. We now estimate, a.e. in $[0, T_{\max})$, the last four integrands in the right–hand side of (6.22). By (6.17) we get

$$k_1 \|u'\|_{H^0}^2 \le 2k_1 \Upsilon. \tag{6.23}$$

Moreover, by the embedding $H^1(\Omega; \Gamma) \hookrightarrow L^2(\Omega) \times L^2(\Gamma)$, there is a positive constant $k_3$, depending only on $\Omega$, such that

$$\|u\|_{H^0}^2 \le k_3 \|u\|_{H^1}^2. \tag{6.24}$$

Consequently, by (6.17), there is a positive constant $k_4$, depending only on $\Omega$, such that

$$k_1 \|u\|_{H^0}^2 \le k_4 \Upsilon. \tag{6.25}$$

To estimate the addendum $2p_1C_{p_1}\int_\Omega \alpha|u|^{p_1-1}|u_t|$ we now distinguish between the cases $p_1=2$ and $p_1>2$. When $p_1=2$, by (6.17), (6.25) and Young inequality,

$$2p_1C_{p_1}\int_\Omega \alpha|u|^{p-1}|u_t| \le p_1C_{p_1}\|\alpha\|_\infty(\|u\|_{H^0}^2+\|u'\|_{H^0}^2)\le k_5\Upsilon, \tag{6.26}$$

where $k_5=2p_1C_{p_1}\|\alpha\|_\infty(1+k_3)$.

When $p_1>2$, for any $\varepsilon\in(0,1]$ to be fixed later, by weighted Young inequality

$$2p_1C_{p_1}\int_\Omega \alpha|u|^{p_1-1}|u_t| \le 2(p_1-1)C_{p_1}\varepsilon^{1-p_1'}\int_\Omega \alpha|u|^{p_1}+2\varepsilon C_{p_1}\int_\Omega\alpha|u_t|^{p_1}. \tag{6.27}$$

By (6.17) we have

$$2(p_1-1)C_{p_1}\varepsilon^{1-p_1'}\int_\Omega \alpha|u|^{p_1}\le 2(p_1-1)\varepsilon^{1-p_1'}\Upsilon. \tag{6.28}$$

Moreover by (1.29) we have $p_1\le m=\overline{m}$ and consequently $|u_t|^{p_1}\le 1+|u_t|^m$ a.e. in $\Omega$, which yields

$$\int_\Omega \alpha|u_t|^{p_1}\le\int_\Omega\alpha+\int_\Omega\alpha|u_t|^m\le\|\alpha\|_\infty|\Omega|+\|[u_t]_\alpha\|_{m,\alpha}^m. \tag{6.29}$$

Plugging (6.28) and (6.29) in (6.27) we get, as $\varepsilon\le 1$,

$$2p_1C_{p_1}\int_\Omega \alpha|u|^{p_1-1}|u_t|\le k_6\left(\varepsilon^{1-p_1'}\Upsilon+\varepsilon\|[u_t]_\alpha\|_{m,\alpha}^m+1\right) \tag{6.30}$$

where $k_6$ is a positive constant independent on $\varepsilon$.

Comparing (6.24) and (6.30) we get that for $p\ge 2$ we have

$$2p_1C_{p_1}\int_\Omega \alpha|u|^{p_1-1}|u_t|\le k_7\left[(1+\varepsilon^{1-p_1'})\Upsilon+\varepsilon\|[u_t]_\alpha\|_{m,\alpha}^m+1\right] \tag{6.31}$$

where $k_7$ is a positive constant independent on $\varepsilon$.

We estimate the last integrand in the right–hand side of (6.22) by transposing from $\Omega$ to $\Gamma_1$ the arguments used to get (6.31). At the end we get

$$2q_1C_{q_1}\int_{\Gamma_1}\beta|u|^{q_1-1}|(u_{|\Gamma})_t|\le k_8\left[(1+\varepsilon^{1-q_1'})\Upsilon+\varepsilon\|[(u_{|\Gamma})_t]_\beta\|_{\mu,\beta,\Gamma_1}^\mu+1\right] \tag{6.32}$$

where $k_8$ is a positive constant independent on $\varepsilon$.

Plugging estimates (6.23), (6.25), (6.31) and (6.32) into (6.22) we get

$$\begin{aligned}\Upsilon(t)\le k_2&+\int_0^t\left[(k_7\varepsilon-c_m')\|[u_t]_\alpha\|_{m,\alpha}^m+(k_8\varepsilon-c_\mu')\|[(u_\Gamma)_t]_\beta\|_{\mu,\beta}^\mu\right]\\&+k_9\int_0^t\left[(1+\varepsilon^{1-p_1'}+\varepsilon^{1-q_1'})\Upsilon+1\right]\qquad\text{for all } t\in[0,T_{\max}).\end{aligned} \tag{6.33}$$

where $k_9$ is a positive constant independent on $\varepsilon$. Fixing $\varepsilon=\varepsilon_1$, where $\varepsilon_1=\min\{1,c_m'/k_7,c_\mu'/k_8\}$, and setting $k_{10}=k_9(1+\varepsilon_1^{1-p'}+\varepsilon_1^{1-q'})$, the estimate (6.33) reads as

$$\Upsilon_u(t)\le\int_0^t k_{10}\,(1+\Upsilon)\qquad\text{for all } t\in[0,T_{\max}).$$

Then, by Gronwall Lemma (see [53, Lemma 4.2, p. 179]), $\Upsilon$ is bounded in $[0,T_{\max})$, hence by (6.17) $\|u\|_{H^1}$ and $\|u'\|_{H^0}$ are bounded in $[0,T_{\max})$, contradicting (6.14). $\square$

**Proof of Theorem 1.6.** It follows by Remarks 3.1, 3.4, 6.4 and Theorem 6.2. $\square$

## Appendix A. On the Cauchy problem for locally Lipschitz perturbations of maximal monotone operators

The aim of this section is to complete the statement of the local existence–uniqueness result in [19] concerning locally Lipschitz perturbations of maximal monotone operators. We first recall it, changing the notation to fit with (4.14).

Let $\mathcal{A}_1 : D(\mathcal{A}_1) \subset \mathcal{H} \to \mathcal{H}$ be a maximal monotone operator on the (real) Hilbert space $\mathcal{H}$, $(\cdot,\cdot)_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ respectively denoting its scalar product and norm. Moreover let $\mathcal{F}_1 : \mathcal{H} \to \mathcal{H}$ be a locally Lipschitz map, i.e. for any $R \geq 0$ there is $L(R) \geq 0$ such that

$$\|\mathcal{F}_1(U) - \mathcal{F}_1(V)\|_{\mathcal{H}} \leq L(R)\,\|U - V\|_{\mathcal{H}} \qquad \text{provided } \|U\|_{\mathcal{H}}, \|V\|_{\mathcal{H}} \leq R\,. \tag{A.1}$$

Given any $h \in L^1_{\text{loc}}([0,\infty);\mathcal{H})$, we are concerned with the Cauchy problem

$$U' + \mathcal{A}_1(U) + \mathcal{F}_1(U) \ni h \qquad \text{in } \mathcal{H}, \qquad U(0) = U_0 \in \mathcal{H}, \tag{A.2}$$

**Theorem A.1** ([19, Theorem 7.2])**.** *Suppose that $\mathcal{A}_1$ is a maximal monotone operator in $\mathcal{H}$ with $0 \in \mathcal{A}_1(0)$ and $\mathcal{F}_1$ satisfies* (A.1)*. Then for any $U_0 \in D(\mathcal{A}_1)$ and $h \in W^{1,1}_{loc}([0,\infty);\mathcal{H})$ problem* (A.2) *has a unique maximal strong solution $U$ in the interval $[0, T_{max})$. Moreover for any $U_0 \in \overline{D(\mathcal{A}_1)}$ and $h \in L^1_{loc}([0,\infty);\mathcal{H})$ problem* (A.2) *has a unique maximal generalized solution in $[0, T_{max})$. In both cases we have $\lim_{t\to T^-_{max}} \|U(t)\|_{\mathcal{H}} = \infty$ provided $T_{max} < \infty$.*

*Remark* A.1. It is well–known that $T_{\max} = \infty$ for any datum $U_0$ when $\mathcal{F}_1$ is globally Lipschitz, i.e. (A.1) holds with $R = \infty$, see [53, Theorems 4.1 and 4.1A].

The aim of this section is to point out the continuous dependence of $U$ from $U_0$ and $h$, which is a standard fact when $\mathcal{F}$ is globally Lipschitz, since the author did not find a precise reference for this fact when $\mathcal{F}$ is only locally Lipschitz. We shall denote by $U = U(U_0, h)$ the maximal generalized solution corresponding to $U_0$ and $h$ and by $T_{\max} = T_{\max}(U_0, h)$ the right–endpoint of its domain.

**Theorem A.2.** *Under the assumptions of Theorem A.1, given $U_0, (U_{0n})_n$ in $\overline{D(\mathcal{A}_1)}$ such that $U_{0n} \to U_0$ in $\mathcal{H}$ and $h_n \to h$ in $L^1_{loc}([0,\infty);\mathcal{H})$, we have*

i) *$T_{max}(U_0, h) \leq \liminf_n T_{max}(U_{0n}, h_n)$, and*
ii) *$U(U_{0n}, h_n) \to U(U_0, h)$ in $C([0, T^*];\mathcal{H})$ for any $T^* \in (0, T_{max}(U_0, h))$.*

*Proof.* The proof is based on the arguments of the proof of Theorem A.1, so we are going to recall some details of it. The solution $U$ is found as the solution of a modified version of (A.2), where $\mathcal{F}_1$ is replaced by a globally Lipschitz map $\mathcal{F}_1^R$ given by

$$\mathcal{F}_1^R(U) = \begin{cases} \mathcal{F}_1(U), & \text{if } \|U\|_{\mathcal{H}} \leq R, \\ \mathcal{F}_1\left(\frac{RU}{\|U\|_{\mathcal{H}}}\right), & \text{if } \|U\|_{\mathcal{H}} \geq R, \end{cases}$$

where $R$ is chosen so that $\|U_0\|_{\mathcal{H}} < R$. Then it is proved that $\mathcal{F}_1^R$ is globally Lipschitz, with Lipschitz constant $L(R)$, and that $\mathcal{A}_1^R = \mathcal{A}_1 + L(R)I + \mathcal{F}_1^R$ is

maximal monotone, hence by [53, Theorem 4.1] the Cauchy problem

$$\text{(A.3)}\qquad \begin{cases} U'+\mathcal{A}_1(U)+\mathcal{F}_1^R(U)=U'+\mathcal{A}_1^R(U)-L(R)U\ni h & \text{in } \mathcal{H},\\ U(0)=U_0\in\mathcal{H}, \end{cases}$$

has a unique generalized solution $U$ in $[0,\infty)$ provided $h\in L^1_{\text{loc}}([0,\infty);\mathcal{H})$ and $U_0\in\overline{D(\mathcal{A}_1)}$, which is actually strong provided $h\in W^{1,1}_{\text{loc}}([0,\infty);\mathcal{H})$ and $U_0\in D(\mathcal{A}_1)$. The existence of a solution of (A.2) in some interval $[0,t^*]$ then follows by choosing $t^*$ (small), depending on $R$ and $h$, such that

$$\text{(A.4)}\qquad \|U(t)\|_{\mathcal{H}}\le R \qquad \text{for all } t\in[0,t^*].$$

Our first claim is that, choosing $R=2(1+\|U_0\|_{\mathcal{H}})$, there is $T_1:[0,\infty)^2\to(0,1]$, decreasing in both variables, such that $t^*=T_1(\|U_0\|_{\mathcal{H}},\|h\|_{L^1(0,1;\mathcal{H})})$ verifies (A.4), so

$$\text{(A.5)}\qquad \|U(t)\|_{C([0,t^*];\mathcal{H})}\le 2(1+\|U_0\|_{\mathcal{H}}).$$

To prove our claim we note, by the same arguments used in the proof of Theorem A.1, that when $U_0\in D(\mathcal{A}_1)$, since $0\in\mathcal{A}_1(0)$, $\mathcal{A}_1^R$ is monotone and $\mathcal{F}_1^R(0)=\mathcal{F}_1(0)$, we have

$$\text{(A.6)}\qquad \frac{d}{dt}\left(\frac12\|U(t)\|^2_{\mathcal{H}}\right)\le L(R)\,\|U(t)\|^2_{\mathcal{H}}+\left(\|h(t)\|_{\mathcal{H}}+\|\mathcal{F}_1(0)\|_{\mathcal{H}}\right)\,\|U(t)\|_{\mathcal{H}}$$

for all $t\in[0,\infty)$, hence by Gronwall Lemma (see [53, Lemma 4.1, p. 179])

$$\text{(A.7)}\qquad \|U(t)\|_{\mathcal{H}}\le e^{L(R)t}\left[\|U_0\|_{\mathcal{H}}+\int_0^t e^{-L(R)s}\Big(\|h(s)\|_{\mathcal{H}}+\|\mathcal{F}_1(0)\|\Big)\,ds\right]$$

for all $t\in[0,\infty)$, which by [53, (4.12), p. 183] holds for all $U_0\in\overline{D(\mathcal{A}_1)}$. By (A.7) then (A.4) holds provided

$$t^*\le 1,\quad e^{L(R)t^*}\le 2,\quad \text{and}\quad e^{L(R)t^*}\left(\|\mathcal{F}_1(0)\|_{\mathcal{H}}+\|h\|_{L^1(0,1;\mathcal{H})}\right)\le 2.$$

Since $L(R)$ in (A.1) can be assumed, without restriction, to be increasing, our claim then follows by setting (where $\log(2/0)$ and $(\log 2)/0$ stand for $\infty$)

$$T_1=\min\left\{1,\frac{\log 2}{L(2+2\,\|U_0\|_{\mathcal{H}})},\frac{1}{L(2+2\,\|U_0\|_{\mathcal{H}})}\log\frac{2}{\|h\|_{L^1(0,1;\mathcal{H})}+\|\mathcal{F}_1(0)\|_{\mathcal{H}}}\right\}.$$

From our first claim then it follows the existence of a maximal generalized solution, as well as its uniqueness, and clearly we have

$$\text{(A.8)}\qquad T_1(\|U_0\|_{\mathcal{H}},\|h\|_{L^1(0,1;\mathcal{H})})<T_{\max}(U_0,h)$$

for all $U_0\in\overline{D(\mathcal{A}_1)}$ and $h\in L^1_{\text{loc}}([0,\infty);\mathcal{H})$.

We now claim that for any $U_0,V_0\in\overline{D(\mathcal{A}_1)}$, $h,k\in L^1_{\text{loc}}([0,\infty);\mathcal{H})$, $M,H$ such that

$$\text{(A.9)}\qquad \max\{\|U_0\|_{\mathcal{H}},\|V_0\|_{\mathcal{H}}\}\le M,\quad \text{and}\quad \max\{\|h\|_{L^1(0,1;\mathcal{H})},\|k\|_{L^1(0,1;\mathcal{H})}\}\le H$$

we have, denoting $U=U(U_0,h)$ and $V=U(V_0,k)$,

$$\text{(A.10)}\qquad \|U(t)-V(t)\|_{\mathcal{H}}\le e^{L(2M+2)t}\left(\|U_0-V_0\|_{\mathcal{H}}+\|h-k\|_{L^1(0,1;\mathcal{H})}\right)$$

for all $t\in[0,T_1(M,H)]$. To prove our claim we note that, being $T_1$ decreasing in both variables, by (A.9) we have

$$\text{(A.11)}\qquad T_1(M,H)\le\min\{T_1(\|U_0\|_{\mathcal{H}},\|h\|_{L^1(0,1;\mathcal{H})}),T_1(\|V_0\|_{\mathcal{H}},\|k\|_{L^1(0,1;\mathcal{H})})\}.$$

Hence, by (A.5) and (A.9), $U$ and $V$ solve in $[0, T_1(M, H)]$ the equation in (A.3) when $R = 2(1 + M)$. Then, first considering data $U_0, V_0 \in D(\mathcal{A}_1)$ and using the monotonicity of $\mathcal{A}_1^R$ we get

$$\frac{d}{dt}\left(\frac{1}{2}\|U - V\|_{\mathcal{H}}^2\right) \le L(2M + 2)\,\|U - V\|_{\mathcal{H}}^2 + \|h - k\|_{\mathcal{H}}\,\|U - V\|_{\mathcal{H}}, \tag{A.12}$$

in $[0, T_1(M, H)]$, hence, by using Gronwall Lemma again

$$\|U(t) - V(t)\|_{\mathcal{H}} \le e^{L(2M+2)t}\left(\|U_0 - V_0\|_{\mathcal{H}} + \int_0^t e^{-L(2M+2)s}\|h(s) - k(s)\|_{\mathcal{H}}\,ds\right),$$

from which, as $T_1 \le 1$, (A.10) follows. By [53, (4.12)] the estimate (A.10) hold for $U_0, V_0 \in \overline{D(\mathcal{A}_1)}$, concluding the proof of our second claim.

Now let $U_0$, $(U_{0n})_n$, $h$, $h_n$ and $T^*$ as in the statement, and denote for shortness $U = U(U_0, h)$, $U_n = U(U_{0n}, h_n)$, $T_{\max} = T_{\max}(U_0, h)$ and $T^n_{\max} = T_{\max}(U_{0n}, h_n)$. We set $M(T^*) = \|U\|_{C([0,T^*];\mathcal{H})}$, $H(T^*) = \|h\|_{L^1(0,T^*+1,\mathcal{H})}$, $T_2(T^*) = T_1(1 + M(T^*), 1 + H(T^*))$ and $\kappa(T^*) \in \mathbb{N}_0$ such that

$$\kappa(T^*)T_2(T^*) < T^* \le [\kappa(T^*) + 1]T_2(T^*) \tag{A.13}$$

i.e. $\kappa(T^*) = \min\{\kappa \in \mathbb{N}_0 : T^*/T_2(T^*) \le \kappa + 1\}$. By (A.9), since $U_{0n} \to U_0$ in $\mathcal{H}$ and $h_n \to h$ in $L^1(0, 1; \mathcal{H})$, there is $n_1(T^*) \in N$ such that $\|U_{0n}\|_{\mathcal{H}} \le M(T^*) + 1$ and $\|h_n\|_{L^1(0,1;\mathcal{H})} \le H(T^*) + 1$ for $n \ge n_1(T^*)$. By the monotonicity of $T_1$ and (A.9) then we have

$$T_2(T^*) \le T_1(\|U_0\|_{\mathcal{H}}, \|h\|_{L^1(0,1;\mathcal{H})}) \quad \text{and} \quad T_2(T^*) \le T_1(\|U_{0n}\|_{\mathcal{H}}, \|h_n\|_{L^1(0,1;\mathcal{H})})$$

for $n \ge n_1(T^*)$. By maximality it follows that

$$T_2(T^*) < T_{\max}, \quad \text{and} \quad T_2(T^*) < T^n_{\max} \quad \text{for } n \ge n_1(T^*). \tag{A.14}$$

By our second claim moreover we have

$$\|U_n - U\|_{C([0,T_2(T^*)];\mathcal{H})} \le e^{L(2M(T^*)+4)T_2(T^*)}\left(\|U_{0n} - U_0\|_{\mathcal{H}} + \|h_n - h\|_{L^1(0,1;\mathcal{H})}\right),$$

from which $U_n \to U$ in $C([0, T_2(T^*)]; \mathcal{H})$, so that

$$U_n(T_2(T^*)) \to U(T_2(T^*)) \quad \text{and} \quad h_n \to h \quad \text{in } L^1(T_2(T^*), T_2(T^*) + 1; \mathcal{H}). \tag{A.15}$$

When $T^* \le T_2(T^*)$, or equivalently $\kappa(T^*) = 0$, the proof of ii) is complete, and by (A.14) we have

$$T^* < T^n_{\max} \qquad \text{for } n \ge n_1(T^*). \tag{A.16}$$

When $T^* > T_2(T^*)$, or equivalently $\kappa(T^*) \ge 1$, we simply repeat previous arguments $\kappa(T^*)$ times, having (A.15) as the starting point. In this way we get that $U_n \to U$ in $C([0, [\kappa(T^*) + 1]T_2(T^*)]; \mathcal{H})$ and $T^* < T^n_{\max}$ for $n \ge n_{\kappa(T^*)+1}(T^*)$. By (A.13) the proof of ii) is then completed, while i) follows, since $T^* \in (0, T_{\max})$ is arbitrary, also using (A.16), concluding the proof. □

## Appendix B. On the Laplace–Beltrami operator

This section is devoted to prove the following result

**Lemma B.1.** *Let $M$ be a $C^2$ compact manifold equipped with a $C^1$ Riemannian metric $(\cdot,\cdot)_M$. Then $-\Delta_M + I$ is a topological and algebraic isomorphism between $W^{s+1,\rho}(M)$ and $W^{s-1,\rho}(M)$ for any $s \in [-1, 1]$ and $1 < \rho < \infty$.*

This fact is well–known when $M$ is smooth (see for example [29], [56] and [58, p.28]). A proof is given in the sequel for the sake of completeness. Due to the linear nature of the problem it is convenient to prove it for Sobolev spaces of complex–valued distributions, since complex interpolation arguments are available. The real case then trivially follows. All the preparatory material in the main body of the paper still hold provided one proceeds as follows. The tangent bundle $T(M)$ is complexified by setting $T(M)^{\mathbb{C}} := \bigcup_{x\in M}\{x\}\times T_x(M)^{\mathbb{C}}$, where $T_x(M)^{\mathbb{C}} \simeq T_x(M)+iT_x(M)$ stands for the complexification of $T_x(M)$ (see [50]). By $\operatorname{Re} v$ and $\operatorname{Im} v$ we shall respectively denote the real and imaginary part of $v\in T(M)^{\mathbb{C}}$. Moreover $(\cdot,\cdot)_M$ is uniquely extended as an hermitian form on $T(M)^{\mathbb{C}}$. Finally $v$ is replaced by $\overline{v}$ in the first integral in (2.2) and (5.4) and in the last one in (2.8)–(2.9).

By repeating the arguments in [42, pp. 38-42] and using the well–known interpolations properties of Sobolev spaces in $\mathbb{R}^n$ (see [59]) one easily proves that

$$W^{s,\rho}(\Gamma)=\begin{cases}(W^{s_0,\rho}(M),W^{s_1,\rho}(M))_{\theta,\rho} & \text{if } s\notin\mathbb{Z},\\ [W^{s_0,\rho}(M),W^{s_1,\rho}(M)]_\theta & \text{if } s\in\mathbb{Z}\end{cases}\tag{B.1}$$

where $s_0,s_1\in\mathbb{Z}$, $s=\theta s_0+(1-\theta)s_1$, $\theta\in(0,1)$, $-2\le s_0\le s_1\le 2$, and $(\cdot,\cdot)_{\theta,\rho}$, $[\cdot,\cdot]_\theta$ respectively denote the real and complex interpolator functors (see [12]).

**Lemma B.2.** *Let $M$ be a $C^2$ compact manifold equipped with a $C^1$ Riemannian metric $(\cdot,\cdot)_M$ and $1<\rho<\infty$. Then*

i) *for any $u\in W^{2,1}(M)$ such that $\Delta_M u\in L^\rho(M)$ we have $u\in W^{2,\rho}(M)$;*
ii) *there is $C=C(\rho,(\cdot,\cdot)_M)>0$ such that*

$$\|u\|_{W^{2,\rho}(M)}\le C\left(\|\Delta_M u\|_{L^\rho(M)}+\|u\|_{L^\rho(M)}\right)\qquad \textit{for all } u\in W^{2,\rho}(M).\tag{B.2}$$

*Proof.* We use the standard localization technique. Since $M$ is compact it posses a finite atlas $\mathcal{U}=\{(U_i,\phi_i), i=1,\dots,r\}$, with $\phi_i(U_i)=B_1$, where $B_R$ denotes the open ball in $\mathbb{R}^n$, $n=\dim M$, of radius $R>0$ centered at the origin. By [54, Theorem 4.1, p. 57] there is a $C^2$ partition of the unity $\mathcal{T}=\{\theta_i, i=1,\dots,r\}$ subordinate to it, i.e. $\theta_i\in C^2(M)$, $0\le\theta_i\le 1$, supp $\theta_i\subset\subset U_i$ for $i=1,\dots,r$, $\sum_{i=1}^r\theta_i=1$ on $M$. In the sequel we shall denote by $C_1,C_2,\dots$ positive constants depending on $\rho$, $(\cdot,\cdot)_M$, $\mathcal{U}$ and $\mathcal{T}$.

We first claim that if $u\in W^{2,s}(M)$ for some $1<s<\rho$ such that $\rho\le sn/(n-s)$ if $s<n$, and $\Delta_M u\in L^\rho(M)$, then $u\in W^{2,\rho}(M)$. To prove our claim we fix $\bar{i}=1,\dots,r$ and we denote $\theta=\theta_{\bar{i}}$, $v=u\theta$, $\widetilde{u}=u\cdot\phi_{\bar{i}}^{-1}\in W^{2,s}(B_1)$, $\widetilde{\theta}=\theta\cdot\phi_{\bar{i}}^{-1}\in C^2(B_1)$, $\widetilde{v}=\widetilde{u}\widetilde{\theta}$. Now set $R\in(0,1)$ such that supp $\widetilde{\theta}\subset\subset B_R$, so that $\widetilde{v}\in W^{2,s}(B_R)$ and supp $\widetilde{v}\subset\subset B_R$. By the expression of $\Delta_M$ in local coordinates we have

$$\Delta_M v=\theta\Delta_M u+2(\nabla_M\theta,\nabla_M u)_M+u\Delta_M\theta.\tag{B.3}$$

Since, by Sobolev embedding theorem, we have $u,|\nabla_M u|_M\in L^\rho(M)$, we get that $\Delta_M v\in L^\rho(M)$. Using its expression in local coordinates again the operator $-\Delta_M$ is expressed, in local coordinates, by $L^2+L^1$, where $L^2=-\partial_i(g^{ij}\partial_j)$ and $L^1=-\frac12 g^{-1}(\partial_j g)g^{ij}\partial_j$, hence by (2.1) and Sobolev embedding theorem we get $L^2\widetilde{v}\in L^\rho(B_R)$. Since, also by (2.1), $L^2$ is a linear uniformly elliptic operator, in the divergence form, with coefficients in $C^1(\overline{B_R})$, we can apply [33, Lemma 2.4.1.4, p. 114] to the homogeneous Dirichlet problem in $B_R$ to conclude that $\widetilde{v}\in W^{2,\rho}(B_1)$,

so that $v \in W^{2\,\rho}(M)$. Summing up for $\bar{i} = 1, \dots, r$ we then get $u \in W^{2,\rho}(M)$, proving our claim. By a reiterate application of previous claim we get i).

To prove ii) we note that, by [2, Theorem 15.2] [16] we get

$$\|\tilde{v}\|_{W^{2,\rho}(B_1)} = \|\tilde{v}\|_{W^{2,\rho}(B_R)} \le C_1 \left(\|\Delta_M \tilde{v}\|_{L^\rho(B_R)} + \|\tilde{v}\|_{L^\rho(B_1)}\right), \tag{B.4}$$

and then, by (B.3),

$$\|\tilde{v}\|_{W^{2,\rho}(B_1)} \le C_2 \left(\|\Delta_M \tilde{u}\|_{L^\rho(B_1)} + \|\tilde{u}\|_{W^{1,\rho}(B_1)}\right),$$

which, by (2.1), yields

$$\|\tilde{v}\|_{W^{2,\rho}(B_1)} \le C_3 \left(\|\Delta_M u\|_{L^\rho(M)} + \|u\|_{W^{1,\rho}(M)}\right).$$

Summing up for $\bar{i} = 1, \dots, r$ we get

$$\|u\|_{W^{2,\rho}(M)} \le C_4 \left(\|\Delta_M u\|_{L^\rho(M)} + \|u\|_{W^{1,\rho}(M)}\right).$$

Since by (B.1) we have $W^{1,\rho}(M) = [W^{2,\rho}(M), L^\rho(M)]_{1/2}$, by interpolation [59, p. 21]) and weighted Young inequalities we get

$$\|u\|_{W^{2,\rho}(M)} \le C_5 \left(\|\Delta_M u\|_{L^\rho(M)} + \|u\|_{L^\rho(M)}\right) \qquad \text{for all } u \in W^{2,\rho}(M). \tag{B.5}$$

We finally set $C = \sup\left\{\frac{\|u\|_{W^{2,\rho}(M)}}{\|\Delta_M u\|_{L^\rho(M)} + \|u\|_{L^\rho(M)}},\, u \in W^{2,\rho}(M) \setminus \{0\}\right\}$, nothing that $C < \infty$ by (B.5) and $C$ is trivially independent on $\mathcal{U}$ and $\mathcal{T}$. □

*Proof of Lemma B.1.* We denote $A_{s,\rho} = -\Delta_M + I : W^{s+1,\rho}(M) \to W^{s-1,\rho}(M)$. By (2.2), (5.2) and (5.5) we have $\langle A_{0,2}\bar{u}, v\rangle_{H^1(M)} = (v,u)_{H^1}$ for all $u, v \in H^1(M)$, so by Riesz–Fréchet theorem, $A_{0,2}$ is an isomorphism.

We now consider the case $s = 1$, starting with $\rho = 2$. By previous remark, for all $h \in L^2(M)$ there is a unique $u \in H^1(M)$ such that $-\Delta_M u + u = h$. Since [57, Theorem 1.3, p. 304–306] trivially extends to $C^2$ manifolds we get $u \in H^2(M)$, hence also $A_{1,2}$ is an isomorphism.

We now consider $\rho \ge 2$. Given $h \in L^\rho(M)$ there is a unique $u \in H^2(M)$ such that $-\Delta_M u + u = h$, and by Lemma B.2 – i) we have $u \in W^{2,\rho}(M)$, hence $A_{1,\rho}$ is an isomorphism when $\rho \ge 2$.

We now take $1 < \rho < 2$ and we consider $A_{1,\rho}$ as an unbounded linear operator in $L^\rho(M)$ with domain $W^{2,\rho}(M)$. Being bounded from $W^{2,\rho}(M)$ to $L^\rho(M)$ by Lemma B.2 –ii), it is a closed operator. We now claim, as in[33, 47], that $-\Delta_M$ is accretive in $L^\rho(M)$, i.e.

$$\operatorname{Re} \int_M -\Delta_M u\, |u|^{\rho-2}\bar{u} \ge 0 \qquad \text{for all } u \in W^{2,\rho}(M). \tag{B.6}$$

We first take $u \in C^2(M)$ and set $u^\star_\varepsilon = (|u|^2 + \varepsilon)^{(\rho-2)/2} u$ for $\varepsilon > 0$. Hence

$$(\nabla_M u, \nabla_M u^\star_\varepsilon)_M = (|u|^2+\varepsilon)^{(\rho-2)/2}|\nabla_M u|^2_M + \tfrac{\rho-2}{2}(|u|^2+\varepsilon)^{(\rho-4)/2}(\nabla_M u, u^2\nabla_M \bar{u}$$
$$+|u|^2\nabla_M u)_M = (|u|^2+\varepsilon)^{(\rho-4)/2}\left[\varepsilon|\nabla_M u|^2_M + \frac{\rho}{2}|\bar{u}\nabla_M u|^2_M + \tfrac{\rho-2}{2}(\bar{u}\nabla_M u, u\nabla_M \bar{u})_M\right]$$

and then, setting $v = \operatorname{Re}(\bar{u}\nabla_M u)$ and $w = \operatorname{Im}(\bar{u}\nabla_M u)$, we get

$$(\nabla_M u, \nabla_M u^\star_\varepsilon)_M = (|u|^2+\varepsilon)^{(\rho-4)/2}\left[\varepsilon|\nabla_M u|^2_M + (\rho-1)|v|^2_M + |w|^2 + i(\rho-2)(v,w)_M\right].$$

[16] or, with a slight variant, [33, Theorem 2.3.3.2, p. 106]

Consequently $\mathrm{Re}(\nabla_M u, \nabla_M u_\varepsilon^\star)_M \geq 0$. By (5.4) then $\mathrm{Re} \int_M -\Delta_M u \; \bar{u_\varepsilon^\star} \geq 0$ for all $\varepsilon > 0$. Since $u_\varepsilon^\star \to |u|^{\rho-2}u$ pointwise in $M$, being uniformly bounded, we can pass to the limit as $\varepsilon \to 0^+$ and get (B.6) for all $u \in C^2(M)$. By density our claim is proved. By (B.6) we immediately get that $\mathrm{Re}\, \langle A_{1,\rho}u, |u|^{\rho-2}\bar{u}\rangle_{L^\rho(M)} \geq \|u\|^\rho_{L^\rho(M)}$ for all $u \in W^{2,\rho}(M)$, from which $A_{1,\rho}$ is injective and, by Hölder inequality,

$$\|u\|_{L^\rho(M)} \leq \|A_{1,\rho}u\|_{L^\rho(M)} \qquad \text{for all } u \in W^{2,\rho}(M),$$

so $\mathrm{Rg}(A_{1,\rho})$ is closed. But $L^2(M) = \mathrm{Rg}(A_{1,2}) \subset \mathrm{Rg}(A_{1,\rho})$, and $L^2(M)$ is dense in $L^\rho(M)$, hence $\mathrm{Rg}(A_{1,\rho})$ is dense, so $\mathrm{Rg}(A_{1,\rho}) = L^\rho(M)$ and $A_{1,\rho}$ is an isomorphism also when $1 < \rho < 2$.

We now consider the case $s = -1$. By (5.2) and (5.4) we have

$$\langle A_{s,\rho}u, v\rangle_{W^{1-s,\rho'}(M)} = \langle A_{-s,\rho'}v, u\rangle_{W^{1+s,\rho}(M)}$$

for all $s \in [-1, 1]$, $u \in W^{s+1,\rho}(M)$ and $v \in W^{1-s,\rho'}(M)$, hence $A_{-1,\rho}$ is the Banach adjoint of $A_{1,\rho'}$. It follows then that $A_{-1,\rho}$ is an isomorphism for $1 < \rho < \infty$. Finally the result holds for $s \in [-1, 1]$ by (B.1) and interpolation theory (see [12]). □

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## References


[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York-London, 1975, Pure and Applied Mathematics, Vol. 65.

[2] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), 623–727.

[3] A. Ambrosetti and G. Prodi, *A primer of nonlinear analysis*, Cambridge University Press, Cambridge, 1993.

[4] K. T. Andrews, K. L. Kuttler, and M. Shillor, *Second order evolution equations with dynamic boundary conditions*, J. Math. Anal. Appl. **197** (1996), no. 3, 781–795.

[5] G. Autuori and P. Pucci, *Kirchhoff systems with dynamic boundary conditions*, Nonlinear Anal. **73** (2010), no. 7, 1952–1965.

[6] ———, *Kirchhoff systems with nonlinear source and boundary damping terms*, Commun. Pure Appl. Anal. **9** (2010), no. 5, 1161–1188.

[7] G. Autuori, P. Pucci, and M. C. Salvatori, *Global nonexistence for nonlinear Kirchhoff systems*, Arch. Ration. Mech. Anal. **196** (2010), no. 2, 489–516.

[8] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff, Amsterdam, 1976.

[9] ———, *Analysis and control of nonlinear infinite-dimensional systems*, Mathematics in Science and Engineering, vol. 190, Academic Press, Inc., Boston, MA, 1993.

[10] ———, *Nonlinear differential equations of monotone types in Banach spaces*, Springer Monographs in Mathematics, Springer, New York, 2010.

[11] JJ. T. Beale, *Spectral properties of an acoustic boundary condition*, Indiana Univ. Math. J. **26** (1976), 199–222.

[12] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin-New York, 1976, Grundlehren der Mathematischen Wissenschaften, No. 223.

[13] L Bociu, *Local and global wellposedness of weak solutions for the wave equation with nonlinear boundary and interior sources and damping*, Nonlinear Anal. (2008).

[14] L. Bociu and I. Lasiecka, *Uniqueness of weak solutions for the semilinear wave equations with supercritical boundary/interior sources and damping*, Discrete Contin. Dyn. Syst. **22** (2008), no. 4, 835–860.

[15] ———, *Local Hadamard well-posedness for nonlinear wave equations with supercritical sources and damping*, J. Differential Equations **249** (2010), no. 3, 654–683.

[16] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011.

[17] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and I. Lasiecka, *Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping—source interaction*, J. Differential Equations **236** (2007), no. 2, 407–459.

[18] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and P. Martinez, *Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term*, J. Differential Equations **203** (2004), no. 1, 119–158.

[19] I. Chueshov, M. Eller, and I. Lasiecka, *On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation*, Comm. Partial Differential Equations **27** (2002), no. 9-10, 1901–1951.

[20] F. Conrad and Ö. Morgül, *On the stabilization of a flexible beam with a tip mass*, SIAM J. Control Optim. **36** (1998), no. 6, 1962–1986 (electronic).

[21] Darmawijoyo and W. T. van Horssen, *On boundary damping for a weakly nonlinear wave equation*, Nonlinear Dynam. **30** (2002), no. 2, 179–191.

[22] E. DiBenedetto, *Real analysis*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston, Inc., Boston, MA, 2002.

[23] G. G. Doronin, N. A. Lar′kin, and A. J. Souza, *A hyperbolic problem with nonlinear second-order boundary damping*, Electron. J. Differential Equations (1998), No. 28, 10 pp. (electronic).

[24] A. Favini, C. G. Gal, G. Ruiz Goldstein, J. A. Goldstein, and S. Romanelli, *The non-autonomous wave equation with general Wentzell boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A **135** (2005), no. 2, 317–329.

[25] A. Figotin and G. Reyes, *Lagrangian variational framework for boundary value problems*, J. Math. Phys. **56** (2015), no. 9, 1 – 35.

[26] N. Fourrier and I. Lasiecka, *Regularity and stability of a wave equation with a strong damping and dynamic boundary conditions*, Evol. Equ. Control Theory **2** (2013), no. 4, 631–667.

[27] C. G. Gal, G. R. Goldstein, and J. A. Goldstein, *Oscillatory boundary conditions for acoustic wave equations*, J. Evol. Equ. **3** (2003), no. 4, 623–635.

[28] V. Georgiev and G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source terms*, J. Differential Equations **109** (1994), no. 2, 295–308.

[29] F. Gesztesy, I. Mitrea, D. Mitrea, and M. Mitrea, *On the nature of the Laplace-Beltrami operator on Lipschitz manifolds*, J. Math. Sci. (N. Y.) **172** (2011), no. 3, 279–346, Problems in mathematical analysis. No. 52.

[30] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer–Verlag, Berlin–New York, 1983.

[31] G. Goldstein R., *Derivation and physical interpretation of general boundary conditions*, Adv. Differential Equations **11** (2006), no. 4, 457–480.

[32] P. J. Graber and I. Lasiecka, *Analyticity and Gevrey class regularity for a strongly damped wave equation with hyperbolic dynamic boundary conditions*, Semigroup Forum **88** (2014), no. 2, 333–365.

[33] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[34] B. Guo and C.-Z. Xu, *On the spectrum-determined growth condition of a vibration cable with a tip mass*, IEEE Trans. Automat. Control **45** (2000), no. 1, 89–93.

[35] E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, Courant Lecture Notes in Mathematics, vol. 5, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[36] T. Kashiwabara, C. M. Colciago, L. Dedè, and A. Quarteroni, *Well-Posedness, Regularity, and Convergence Analysis of the Finite Element Approximation of a Generalized Robin Boundary Value Problem*, SIAM J. Numer. Anal. **53** (2015), no. 1, 105–126.

[37] I. Lasiecka and D. Tataru, *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential Integral Equation **6** (1993), no. 3, 507–533.

[38] H. A. Levine, P. Pucci, and J. Serrin, *Some remarks on global nonexistence for nonautonomous abstract evolution equations*, Harmonic Analysis and Nonlinear Differential Equations (M. L. Lapidus, L. H. Harper, and A. J. Rumbos, eds.), Contemp. Math., **208**, American Mathematical Society, 1997, pp. 253–263.

[39] H. A. Levine and J. Serrin, *Global nonexistence theorems for quasilinear evolution equations with dissipation*, Arch. Rational Mech. Anal. **137** (1997), no. 4, 341–361.
[40] J. L. Lions, *Lectures on elliptic partial differential equations*, Tata Institute of Fundamental Research, Mumbai, India, 1957, http://www.math.tifr.res.in/∼publ/ln/tifr10.pdf.
[41] J.-L. Lions and E. Magenes, *Problemi ai limiti non omogenei. III*, Ann. Scuola Norm. Sup. Pisa (3) **15** (1961), 41–103, italian.
[42] ______, *Problèmes aux limites non homogènes et applications. Vol. 1*, Travaux et Recherches Mathématiques, No. 17, Dunod, Paris, 1968.
[43] M. Marcus and V. J. Mizel, *Absolute continuity on tracks and mappings of Sobolev spaces*, Arch. Rational Mech. Anal. **45** (1972), 294–320.
[44] Ö. Morgül, B. Rao, and F. Conrad, *On the stabilization of a cable with a tip mass*, IEEE Trans. Automat. Control **39** (1994), no. 10, 2140–2145.
[45] P.M.C. Morse and K.U. Ingard, *Theoretical acoustics*, International series in pure and applied physics, Princeton University Press, 1968.
[46] D. Mugnolo, *Damped wave equations with dynamic boundary conditions*, J. Appl. Anal. **17** (2011), no. 2, 241–275.
[47] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.
[48] P. Pucci and J. Serrin, *Global nonexistence for abstract evolution equations with positive initial energy*, J. Differential Equations **150** (1998), no. 1, 203–214.
[49] P. Radu, *Weak solutions to the Cauchy problem of a semilinear wave equation with damping and source terms*, Adv. Differential Equations **10** (2005), no. 11, 1261–1300.
[50] S. Roman, *Advanced linear algebra*, third ed., Graduate Texts in Mathematics, vol. 135, Springer, New York, 2008.
[51] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.
[52] J. Serrin, G. Todorova, and E. Vitillaro, *Existence for a nonlinear wave equation with damping and source terms*, Differential Integral Equations **16** (2003), no. 1, 13–50.
[53] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997.
[54] S. Sternberg, *Lectures on differential geometry*, second ed., Chelsea Publishing Co., New York, 1983, With an appendix by Sternberg and Victor W. Guillemin.
[55] W. A. Strauss, *On continuity of functions with values in various Banach spaces*, Pacific J. Math. **19** (1966), 543–551.
[56] R. S. Strichartz, *Analysis of the Laplacian on the complete Riemannian manifold*, J. Funct. Anal. **52** (1983), no. 1, 48–79.
[57] M. E. Taylor, *Partial differential equations*, Texts in Applied Mathematics, vol. 23, Springer-Verlag, New York, 1996, Basic theory.
[58] ______, *Partial differential equations. III*, Applied Mathematical Sciences, vol. 117, Springer-Verlag, New York, 1997.
[59] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, 1978.
[60] ______, *Theory of function spaces. II*, Monographs in Mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992.
[61] J. L. Vazquez and E. Vitillaro, *Wave equation with second-order non-standard dynamical boundary conditions*, Math. Models Methods Appl. Sci. **18** (2008), no. 12, 2019–2054.
[62] ______, *Heat equation with dynamical boundary conditions of reactive-diffusive type*, J. Differential Equations **250** (2011), no. 4, 2143–2161.
[63] E. Vitillaro, *Global nonexistence theorems for a class of evolution equations with dissipation and application*, Arch. Rational Mech. Anal. **149** (1999), 155–182.
[64] ______, *Some new results on global nonexistence and blow-up for evolution problems with positive initial energy*, Rend. Istit. Mat. Univ. Trieste **31** (2000), no. suppl. 2, 245–275.
[65] ______, *Global existence for the wave equation with nonlinear boundary damping and source terms*, J. Differential Equations **186** (2002), no. 1, 259–298.
[66] ______, *Strong solutions for the wave equation with a kinetic boundary condition*, Recent trends in nonlinear partial differential equations. I. Evolution problems, Contemp. Math., vol. 594, Amer. Math. Soc., Providence, RI, 2013, pp. 295–307.

[67] T.-J. Xiao and L. Jin, *Complete second order differential equations in Banach spaces with dynamic boundary conditions*, J. Differential Equations **200** (2004), no. 1, 105–136.
[68] T.-J. Xiao and J. Liang, *Second order parabolic equations in Banach spaces with dynamic boundary conditions*, Trans. Amer. Math. Soc. **356** (2004), no. 12, 4787–4809.

(E. Vitillaro) Dipartimento di Matematica ed Informatica, Università di Perugia, Via Vanvitelli,1 06123 Perugia ITALY

*E-mail address*: enzo.vitillaro@unipg.it